BULLETIN

OF THE

AMERICAN MATHEMATICAL SOCIETY

A HISTORICAL AND CRITICAL REVIEW
OF MATHEMATICAL SCIENCE

EDITED BY

EARLE RAYMOND HEDRICK
WALLIE ABRAHAM HURWITZ JOHN WESLEY YOUNG

WITH THE ASSISTANCE OF

ROBERT WILBUR BURGESS DERRICK NORMAN LEHMER
DUNHAM JACKSON TULLIO LEVI-CIVITA
EDWARD KASNER HENRY LEWIS RIETZ

VOLUME XXIX, NUMBER 6

JUNE, 1923

PUBLISHED BY THE SOCIETY
LANCASTER, PA., AND NEW YORK

1923

Entered at the Post Office at Lancaster, Pa., as second class matter
PUBLISHED MONTHLY, EXCEPT AUGUST AND SEPTEMBER

Whole No. 315 $7.00 a Year

Transformations of Surfaces

by

LUTHER PFAHLER EISENHART

Professor of Mathematics in Princeton University.

A continuation of the author's

Differential Geometry

Published with the cooperation of

The National Research Council

380 pages. Price, $4.00

Princeton University Press

Princeton, N. J.

The Cambridge Colloquium, Part II

ANALYSIS SITUS

By

OSWALD VEBLEN

Published by the Society, May, 1922. 8 + 150 pages, octavo. Paper covers. Price $2.00; to members of the Society, $1.50.

The Cambridge Colloquium, Parts I and II, bound together in cloth. Price $3.50; to members of the Society, $3.00.

Part I separately (Functionals and their Applications, by G. C. Evans, 1918, 12 +136 pages, paper.) $2.00; to members of the Society, $1.50.

Address all orders to

AMERICAN MATHEMATICAL SOCIETY,
501 West 116th St., New York City.

INCORPORATION OF THE AMERICAN MATHEMATICAL SOCIETY

At the February, 1921, meeting of the Council, Professors H. W. Tyler, W. F. Osgood, and D. E. Smith were appointed a committee with instructions to incorporate the Society and to procure a national charter if that were feasible. A bill to incorporate in the District of Columbia was introduced into the Senate and the House of Representatives, but it met with considerable opposition in the committee of the Senate. At the end of the last session of Congress the committee decided to abandon the idea of a special bill and to incorporate under the Code of the District of Columbia.

On May 3, 1923, incorporation was effected under the Code of the District of Columbia, the incorporators being H. W. Tyler, H. L. Hodgkins, Harry English, O. S. Adams, and J. T. Erwin.

The certificate of incorporation is very simple and contains beside the preamble only the following four articles.

I. The name or title by which this Society shall be known in law is the AMERICAN MATHEMATICAL SOCIETY.

II. The term for which it is organized shall be perpetual.

III. The particular business and objects of the Society are the furtherance of the interests of mathematical scholarship and research.

In furtherance of and not in limitation of the general objects of the Society and the powers conferred by the laws of the District of Columbia, it is hereby expressly provided that the following persons shall be eligible for membership in this corporation, to wit: the members now forming the unincorporated association of the same name and such other persons as may from time to time be elected to membership under the by-laws to be hereafter enacted. The corporation hereby constituted shall have power to fill all vacancies created by death, resignation, or otherwise, to provide for the election of foreign, domestic, or honorary associate members, and the division of such members into classes, and to make all needed rules and regulations for the purpose of carrying out its proper corporate objects.

IV. The number of its trustees, directors or managers for the first year of this corporation's existence shall be thirty-one (31).

The first meeting of the Corporation was held at the Cosmos Club, Washington, D. C., on May 4, 1923, at 12:45 p. m. All the incorporators were present and they elected as their associates for the first year thirty-one members of the Council for 1921. The next meeting of the Corporation will take place at Poughkeepsie in connection with the summer meeting. It is expected that at that time the incorporated body will assume control of the affairs of the Society and continue the objects for which the old Society was organized.

R. G. D. RICHARDSON,
Secretary.

REPORT ON CURVES TRACED ON ALGEBRAIC SURFACES*

BY S. LEFSCHETZ

1. *Introduction.* An extensive literature to which we propose to devote this report has grown up around the following question and the related transcendental theory. *Given an algebraic surface, what can be said in regard to the distribution of its continuous systems of algebraic curves?*† For various reasons we have chosen in place of the chronological presentation one in which function theory and analysis situs play the predominant part, and that has been made possible by two papers of Poincaré (*s*, I, II). We must however recall at the outset that the general answer to the above question had been given earlier by Severi (*u*, V, VI), his methods being largely of an algebro-geometric nature (see § 14), except for the use of a very important transcendental theorem due to Picard (*q*, II, p. 241). In favor of the methods which dominate this report, it must be stated that they alone made possible the solution of some important problems, and furthermore have notably enriched the theory.

A question similar to the above may be asked concerning algebraic varieties, but in order to remain within proper bounds, we have deemed it best to omit them altogether.

2. *Connectivity Indices.* We shall have occasion to consider throughout a basic n-dimensional manifold W_n,‡ usually an algebraic curve ($n = 2$), or a surface ($n = 4$). A sum of closed, k-dimensional, two-sided, analytic manifolds in W_n is called a *k-cycle* of the manifold, and shall be denoted by Γ_k. If it bounds, it is a zero-cycle; the fact being expressed by a homology: $\Gamma_k \sim 0 \bmod W_n$. Homologies can be added

* Presented before the Society at the Symposium held in Chicago, April 13, 1923.

† The properties of linear and continuous systems, so successfully investigated by Castelnuovo, Enriques, and Severi, have been described by them in three readily accessible reports (*q*, II, p. 485; *e*; *u*, XI). These and similar references refer to the bibliography at the end of the report.

‡ For a more extended topological discussion see *n*, II, § 1; *v*, Ch. 4.

and also multiplied (but not divided) by integers. If several k-cycles satisfy no homology they are *independent.* The maximum number R_k of such cycles is the kth *connectivity index* (*linear index* for $k = 1$). The index R_1 of an algebraic variety is always even. For an algebraic curve, $\frac{1}{2}R_1 = p$ is the *genus,* for an algebraic surface or variety, $\frac{1}{2}R_1 = q$ is the *irregularity* (n, II, p. 233; n, VII, Ch. 2).

3. *Abelian Integrals.* We recall that by an *Abelian integral* attached to the algebraic curve C, of order m and genus p, whose equation is $f(x, y) = 0$, is meant an integral

$$\int R(x, y)dx,$$

where R is rational in the coordinates of a variable point on C. A *period* of it is its value taken over a cycle Γ_1 of C. The integral is of the *first kind* when it is holomorphic everywhere on the curve. It is then of the form

$$\int \frac{Q(x, y)dx}{f_y'},$$

where Q is an adjoint polynomial of order $m - 3$.* The maximum number of linearly independent † integrals of the first kind is equal to p (see r). Let $u_1, u_2, \cdots, u_p$ be a set of such integrals, and consider the equations

$$\sum_{h=1}^{n} \int_A^{A_h} du_i = v_i, \qquad (i = 1, 2, \cdots, p),$$

where the v's are constant, and the unknowns are the upper limits A_h on C.

(a) When there is more than one solution there are an infinite number, and the most general one depends linearly upon $r \geqq n - p$ parameters (Abel; Clebsch; c, p. 395). Their totality constitutes a so-called *linear series* on C.

(b) For $n = p$ the solution is in general *unique* (inversion in the sense of Jacobi; see c, p. 463). In the exceptional case, a certain number $p - p'$ of the points may be assigned arbitrarily, and the remaining p' are then uniquely determined.

* When C has no other singularities than double points with distinct tangents, an adjoint polynomial is one which vanishes at the double points. A similar definition holds for surfaces, with the double curve in place of the double points. See r.

† In the sense that no linear combination of them is constant.

4. *Algebraic Surfaces.* We shall denote our surface by F, take for its equation $f(x, y, z) = 0$ and assume it irreducible, of order m, with the genus of a generic plane section (to be denoted by H) equal to p. The special sections cut out by planes $y = Ct$ shall be called H_y. We shall furthermore assume F with only ordinary singularities * (double curve with triple points in finite number) and in arbitrary position as to the axes and the plane at infinity. The section H_y will then be of genus p unless its plane is tangent to F, when the genus is lowered to $p - 1$. The values $a_1, a_2, \cdots, a_n$ corresponding to contact are its *critical* values and n is the class of F. To each a_k is attached an important cycle Γ_1 of F: it is the cycle δ_k which tends to a point when y approaches a_k (vanishing cycle). When y turns around a_k, the increment of any Γ_1 of H_y is homologous to a multiple of δ_k mod. H_y (Picard, *q*, I, p. 95; *n*, VII, Ch. 2).

5. *Abelian Sums attached to Algebraic Curves.* A notable achievement it was of Poincaré's to have discovered a set of simple expressions that characterize continuous systems of algebraic curves on F (*s*, II, p. 56). By taking advantage of certain results known before his work but which follow readily also from his methods, we shall obtain easily a set of somewhat simpler expressions that will suffice for our purposes (*n*, IV, p. 343; *u*, IX, p. 204).

To H_y are attached p linearly independent integrals of the first kind of the particularly simple type

$$u_j = \int \frac{Q_j(x, y, z)}{f_z'}\, dx \qquad (j = 1, 2, \cdots, p),$$

where the Q's are adjoint polynomials of order $m - 3$ in x and z; however q of them, say the last q, are of degree $m - 2$ in x, y, and z, while the remaining $p - q$ of them are still of degree $m - 3$ when y is taken into consideration. Furthermore the periods of u_{p-q+l} with respect to the vanishing cycles δ_k are all zero.†

* Any F can be birationally reduced to that particular kind (Beppo Levi, *o*; Chisini, *f*).

† The integrals u_{p-q+l} are what q linearly independent integrals of total

Let now $A_1, A_2, \cdots, A_m$ be the fixed points of H_y (all at infinity), and let C be an arbitrary algebraic curve on F. The curve C may go through some A's with varied multiplicities. Let $B_1, B_2, \cdots, B_p$ be a group of points on H_y, including all variable points of intersection of C and H_y, and some of the A's, each taken with any multiplicity whatever. Let us then consider the sums

$$v_j(y) = \sum_{k=1}^{p} \int_{A_1}^{B_k} du_j, \qquad (j = 1, 2, \cdots, p). \tag{1}$$

From the behavior of the v's, we find the relations

$$\begin{cases} v_j(y) = \sum \dfrac{\lambda_k}{2\pi i} \displaystyle\int_A^{A_k} \frac{\Omega_{jk}(Y)dY}{Y - y}, & (j = 1, 2, \cdots, p - q), \\ v_{p-q+l}(y) = \alpha_l, & (l = 1, 2, \cdots, q), \end{cases} \tag{2}$$

where the α's are constants, the λ's are integers, and Ω_{jk} is the period of u_j with respect to δ_k. These expressions are basic for what follows.

6. *Existence Theorem and its Interpretation. Under what circumstances can the v's given by* (2) *yield, by means of* (1), *points B_k with an algebraic curve C for locus*? Assign to all but $p' \geqq p$ of the B's the position A_1, and solve for the remainder as in § 3, p' having the same meaning as there. On discussing the equations obtained (Poincaré, *s*, II, p. 75; *s*, III, p. 41; Lefschetz, *n*, VII, Ch. 4; Severi, *u*, IX, p. 278), the following necessary and sufficient conditions are obtained:

(*a*) The v's must be regular at $y = a$.

(*b*) Let there be constants β_k such that

$$\frac{1}{y - y_0} \sum \beta_k Q_k(x, y, z) \equiv Q(x, y, z) \tag{3}$$

is finite when y approaches y_0. Similarly, the expression

$$\frac{1}{y - y_0} \sum \beta_k v_k(y)$$

must equally remain finite when y approaches y_0.

differentials of the first kind of F (see § 11) become when y is held constant. The possibility of subdividing the u's into two groups as here indicated is based upon the fundamental result recalled (loc. cit.), together with a theorem proved by Picard (*q*, II, p. 437) and Severi (*u*, VIII).

Conditions (a) and (b) are unsatisfactory notably in that they involve the particular variable y. From them may be derived others of a more significant nature and in particular not involving y (Lefschetz, n, III; n, IV, p. 345; n, VII, Ch. 4). The polynomial Q in (3) is adjoint of order $m-4$, hence

$$\int\int \frac{Q(x, y, z)}{f_z'}\, dxdy \tag{4}$$

is finite everywhere on F. In the usual terminology we say that it is of the first kind (Noether, p, I). As a matter of fact there is a relation (3) for every Q, hence (4) is arbitrary of its kind. We then have the following theorem.

In order that there may correspond to the (v)*'s an algebraic curve* C*, there must exist a cycle* Γ_2 *with respect to which the period of every double integral of the first kind is zero.**

The cycle Γ_2 is readily described. Let Δ_k be the locus of δ_k as y describes aa_k. Picard (q, II, p. 335) and Poincaré (s, II, p. 57) for finite cycles and Lefschetz (n, IV, Ch. 3) for any cycle, have established the homology

$$\Gamma_2 \sim \sum \lambda_k \Delta_k + \text{part of } H_a, \text{ mod } F.$$

When C exists we have (n, IV, Ch. 4)

$$C \sim \sum \lambda_k \Delta_k + \text{part of } H_a, \text{ mod } F,$$

the λ's being the same integers as in (2). This brings out clearly the relation between them and the curve.

7. *Equivalence of Curves.* A variable curve C of intersection of F with a surface whose equation contains r linear parameters gives rise to a *linear system of dimension* r, denoted by $|C|$. This system is *complete* if its curves do not belong to another of dimension $> r$, as shall be assumed throughout. The complete system determined by a given curve is unique.

If the irregularity q is greater than zero, the surface contains continuous systems whose curves do not all belong to one and the same linear system (e, p. 707). We denote by

* A noteworthy feature of this theorem is that it is the only one known concerning periods of double integrals of the first kind. This is in striking contrast with Abelian integrals of the first kind for the periods of which an extensive body of theorems has long been available.

$\{C\}$ the amplest system generated by all variable curves that may approach C. It contains all systems $|C|$ determined by its curves, and may be thought of as an algebraic variety whose elements are linear systems. Indeed when $\{C\}$ is sufficiently general, this variety is an Abelian variety of genus q, the so-called Picard variety of F.

By $|C+D|$ and $\{C+D\}$ are meant the systems defined by the composite curve $C+D$. They are uniquely determined by $|C|$ and $|D|$ in the first case, and by $\{C\}$ and $\{D\}$ in the second case. If D is in $|C|$ (or in $\{C\}$) we write $|2C|$ (or $\{2C\}$) for the sum. The meaning of $|kC|$ and of $\{kC\}$ follows immediately, if k is a positive integer. Likewise the meaning of $|C-D|$ and of $\{C-D\}$ is immediate (see h). The last two systems may not actually exist; they are then called *virtual.** Although not represented by any geometric configuration, there are important related symbols, and the introduction of these systems (Severi, u, X) has been very useful. Similar remarks hold for $|k_1C_1+k_2C_2+\cdots+k_\rho C_\rho|$ and $\{k_1C_1+k_2C_2+\cdots+k_\rho C_\rho\}$, where the k's are integers.

The v's are the same for all curves of a linear system; and for curves of a continuous system, they differ in the constants α, but not in the integers λ. In any case it is clear that the v's of the same index are combined like their curves.†

A partial result of Poincaré's (s, II, p. 98) which I have completed is the following theorem (n, IV, Ch. 4).

If C and D have the same v's, then D belongs to $|C|$. If their integers λ are the same, i.e., if only the first $p-q$ functions v are the same, then there is a positive integer k such that $\{C+kH\}$ and $\{D+kH\}$ coincide.

This justifies the following definition: C and D are *equivalent*, and we write $C=D$ if there is an E such that $\{C+E\}$ and $\{D+E\}$ coincide. A meaning is then readily attributed

* Moreover, even when $\{C-D\}$ exists it is not necessarily unique.

† Henceforth we mean by the set of v's corresponding to C that obtained when all points of intersection of C with H_ν are considered and no others.

to a relation

$$\mu_1 C_1 + \mu_2 C_2 + \cdots + \mu_\rho C_\rho = 0.$$

These notions go back essentially to Severi (*u*, V, p. 198). However, his definition of equivalence is somewhat narrow. The question has been searchingly examined by Albanese and our definition corresponds to what he calls *virtual equivalence* (*b*, p. 165).

8. *Identity with Homology.* Much interest is added to what precedes by this proposition (Lefschetz, *n*, IV, p. 347; *n*, VII, Ch. 4): *There is complete equivalence between the two relations*

$$\sum \mu_i C_i = 0; \qquad \sum \mu_i C_i \sim 0, \text{ mod } F.$$

Thus a purely algebro-geometric notion is reduced to one of analysis situs. Well known propositions from the latter theory yield at once these results, whose geometric content we owe to Severi (*n*, IV; *n*, VI; *u*, V; *u*, VI):

(*a*) *There is a positive integer ρ such that any $\rho + 1$ curves satisfy a relation of equivalence, this not being the case for some sets of ρ of them.* The number ρ is the well known Picard number (*q*, II, p. 241).

(*b*) *The number ρ is the maximum number of distinct two-cycles with respect to which every double integral of the first kind has a zero period.*

(*c*) *Given a curve C, of sufficiently general type,* there may exist $\sigma - 1$ others, $C_2, C_3, \cdots, C_\sigma$, such that $C_i \not\equiv C_1$, $\lambda C_i = \lambda C_1$, $\lambda > 1$.* The number σ is the product of Poincaré's so-called *torsion coefficients* for linear or two-cycles † (they are the same for F).

(*d*) *There exists an ordinary base for the curves of F, i.e., a set of curves $C_1, C_2, \cdots, C_\rho$, such that whatever C we may write:*

$$\mu C = \mu_1 C_1 + \mu_2 C_2 + \cdots + \mu_\rho C_\rho.$$

The term *ordinary* base was introduced by Severi. By increasing the number of curves to $\rho + \sigma - 1$, we may have what Severi terms a *minimum base*, i.e., such that $\mu = 1$ for

* If virtual curves are admitted no restrictions are needed.

† Examples of surfaces with $\sigma > 1$ have been given by Severi (*u*, VI), Godeaux (*k*) and the writer (*n*, IV, p. 362). On such an F can be found a Γ_1 such that $\lambda\Gamma_1$ bounds ($\lambda > 1$) while Γ_1 does not, and similarly for two-cycles.

every C. Let these curves be $D_1, D_2, \cdots, D_{\rho+\sigma-1}$. Then

$$C = \nu_1 D_1 + \nu_2 D_2 + \cdots + \nu_{\rho+\sigma-1} D_{\rho+\sigma-1}.$$

All these curves may be actually chosen *effective* (non-virtual) except for $\rho = 1$, when it may be necessary to take $\sigma + 1$ curves in the minimum base, if it be desired to have them all effective. The geometric content of this answers completely the question proposed at the beginning of our report. Indeed, let $D_1, D_2, \cdots, D_\nu$ be a minimum base composed exclusively of effective curves. For every C we may write

$$C = \sum \mu_i D_i = \sum \mu_i'' D_i - \sum \mu_i' D_i,$$

where the μ''s and μ'''s are non-negative integers. Then there exists a positive integer μ such that the continuous systems

$$\{C + \mu H + \sum \mu_i' D_i\}, \qquad \{\mu H + \sum \mu_i'' D_i\}$$

coincide (Albanese, *b*, p. 204).

9. *Integrals of the Second and Third Kinds.* Abelian integrals constitute the natural analytical tool in the study of sets of finite points on an algebraic curve, for these points appear either as the set of singular points of the integral, or as the zeros of the integrand. For an entirely similar reason, the integrals that generalize Abelian integrals are of paramount interest in investigations on algebraic curves of a surface, and thus find a proper place here. We recall that an Abelian integral of the *second kind* is one behaving everywhere on its curve C like a rational function. Finally an integral is of the *third kind* if not of one of the other two kinds. Its singularities other than poles consist in a finite number of logarithmic points and to each belongs a logarithmic period corresponding to a small circuit surrounding the point. The sum of the logarithmic periods is zero. Hence there must be at least two logarithmic points, and in fact there is an integral having any two points of C for logarithmic singularities and no other. Integrals of the second kind are linearly independent if no linear combination of them reduces to a rational function. The maximum number of such integrals is twice the genus of C (see *r*).

10. *Generalization to Double Integrals.* On passing to F there are two modes of extending Abelian integrals, first to double integrals, next and much less obviously to integrals of total differentials (Picard, *q*, I, p. 102), which we shall consider below.

We have defined double integrals of the first kind in § 6. On the transcendental side there is no other theorem than the one mentioned there, although the number of linearly independent integrals and the related linear system cut out by adjoint polynomials of order $m - 4$ (canonical system) have played a large part in investigations on surfaces (*e*, p. 704), this being due of course to their invariance under birational transformations. A question which I have been able to settle only in some special cases (*n*, IV, p. 349) is still outstanding: *Can a double integral of the first kind be without periods?* * The answer (which is probably negative) would have an important bearing upon our subject.

In the several existing treatments of double integrals of the second kind (Picard, *q*, II; Lefschetz, *n*, II, p. 242; *n*, VII, Note I), the type

$$\int\int\left(\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}\right)dxdy\,, \qquad (U,V \text{ rational}), \tag{5}$$

introduced by Picard, plays somewhat the same part as that played by rational functions for Abelian integrals. The reason is here again invariance under birational transformations. The following mode of attack (*n*, VII, Note I) seems shortest and best, especially in that it exhibits a noteworthy theorem and is readily extended to integrals of any multiplicity of a higher variety: We take two-cycles in maximum number, say $\Gamma_2^1, \Gamma_2^2, \cdots, \Gamma_2^{\rho_0}$ such that no matter how ample a set of curves $D_1, D_2, \cdots, D_r$ is given, there is a cycle $\sim \Gamma_2^i$ not intersecting any of them. The curves $C_1, C_2, \cdots, C_\rho$ of a Severi base are two-cycles independent of the Γ_2^i, and every Γ_2 depends upon the $\rho_0 + \rho$ thus obtained, whence

$$\rho_0 = R_2 - \rho.$$

It is then found that J is without period with respect

* It will be recalled that the corresponding theorem for Abelian integrals is proved with ease (see *r*).

to Γ_2^i, while on the contrary any integral *not* of this type will have a period with respect to some such cycle, whence it follows that the maximum number of linearly independent integrals of the second kind is ρ_0. But it is known (Alexander, *a*; Lefschetz, *n*, II, p. 239; *n*, VII, Ch. 3) that

$$R_2 = I + 4q + 2,$$

where $I = n - m - 4p$ is the well known Zeuthen-Segre invariant. From this follows Picard's formula

$$\rho_0 = I + 4q - \rho + 2.$$

The very suggestive relation between the number of linearly independent integrals of the second kind and the number of cycles of a certain type may be extended to multiple integrals of higher varieties (*n*, VI).*

11. *Integrals of Total Differentials.* By *integral of total differentials* is meant one of type

$$\int Rdx + Sdy, \qquad \frac{\partial R}{\partial y} = \frac{\partial S}{\partial x}.$$

The classification and independence theorems are as for Abelian integrals with q in place of p (Picard, *q*, I, Ch. V; Castelnuovo-Enriques and Severi, *e*, p. 715; Poincaré, *s*, II, p. 91). The extension of Abel's theorem has been the object of extensive investigations by Severi (*u*, II, III).

We are particularly interested in integrals of the third kind. Let J be one, $C_1, C_2, \cdots, C_k$ its singular curves. In the vicinity of C_i the integral behaves either like a rational function or like a logarithm. In the latter case there is a logarithmic period, and C_i is a logarithmic curve. By investigating the integral which J determines on H_y, Picard has shown that when the logarithmic curves are arbitrary, J exists, provided that their number exceeds a certain integer ρ whose first appearance in the literature was precisely in this connection (*q*, II, p. 240). That it coincides with the integer denoted by ρ in § 8 follows from the following elegant theorem due to Severi.

* As a matter of fact it holds for integrals of the second kind of all types, down to Abelian integrals.

In order that $C_1, C_2, \cdots, C_k$ be logarithmic curves of some integral of total differentials, it is necessary and sufficient that they satisfy a relation of equivalence (Severi, *u*, V, p. 209; Lefschetz, *n*, VII, Note I). From this and Picard's result follows our assertion as to ρ.

12. *Genera of Curves. Number of Intersections.* We shall use the following notations (*u*, V; *n*, I):

$[C]$ = genus of the generic curve of $\{C\}$;
$[CD]$ = number of intersections of C and D;
$[C^2]$ = number of intersections of two curves of $\{C\}$.

An extensive symbolic calculus may be developed for these expressions (*n*, I), based on the following two formulas:

$$[(C+D)^2] = [C^2] + 2[CD] + [D^2], \tag{6}$$

$$[(C+D)] = [C] + [D] + [CD] - 1. \tag{7}$$

The proof of (6) is immediate, and that of (7) (Noether, *p*, III; Picard, *q*, II, p. 106) may be carried out very simply as follows. Subdivide a Riemann surface for C into α_2 two-cells with α_1 edges and α_0 vertices. The expression $\alpha_0 - \alpha_1 + \alpha_2$ is independent of the mode of subdivision and when C is irreducible its value is $2 - 2[C]$. Let now C vary and acquire a new double point. With a properly chosen subdivision, it is found that α_1 alone varies, and in fact decreases by 2, so that $\alpha_0 - \alpha_1 + \alpha_2$ is increased by 2. Similarly, if C acquires d double points, the expression increases by $2d$. Let then the generic curves of $\{C\}$, $\{D\}$, $\{C+D\}$ be irreducible, which is the only case of interest, and let α_i', α_i'' be the α_i of D and of the generic curve of $\{C+D\}$. We have at once

$$2[CD] + \alpha_0'' - \alpha_1'' + \alpha_2'' = (\alpha_0 - \alpha_1 + \alpha_2) + (\alpha_0' - \alpha_1' + \alpha_2'),$$

$$2[CD] + 2 - 2[(C+D)] = 2 - 2[C] + 2 - 2[D],$$

whence (7) follows.

The formulas (6) and (7) can be greatly generalized, and in particular lead to a meaning for the characters of any system whatever, even reducible or virtual. More important,

however, is the following fact. Let $C_1, C_2, \cdots, C_\rho$ be a base, and let

$$\lambda C = \sum \lambda_i C_i, \qquad \mu D = \sum \mu_i C_i;$$

then (n, I, p. 207)

$$(8) \qquad [C] = \prod_{i=1} (1 + C_i)^{\frac{\lambda i}{\lambda}} - \frac{1}{\lambda} \sum_{i=1} \lambda_i,$$

where the product at the right is to be expanded according to the binomial theorem, the terms of degree one and two alone being kept and then replaced by the corresponding character. Similarly (Severi, u, V, p. 223)

$$(9) \quad [CD] = \left[\left(\sum \frac{\lambda_i}{\lambda} G_i \right) \left(\sum \frac{\mu_k}{\mu} C_k \right) \right] = \frac{1}{\lambda\mu} \sum \lambda_i \mu_k [C_i C_k].$$

It is clear that (8) and (9) give the genera and the number of intersections of curves in terms of similar data concerning the curves of the base, when their expressions in terms of the base are known. Let, for example, F be a ruled surface, and denote a generator by G. From analysis situs (n, VII, Ch. 3) it follows readily that any $\Gamma_2 \sim \alpha H + \beta G$. In particular any algebraic curve $C \sim \alpha H + \beta G$, and therefore

$$C = \alpha H + \beta G.$$

(See also Severi, u, I, p. 22.) Since $[G_2] = 0$ we have at once $\alpha = [CG]$, number of intersections of C with any G. Then from $[H^2] = m$, $[HG] = 1$, follows $\beta = \mu - m\alpha$, where $\mu = [CH]$, order of C. In terms of α and μ we have $G = \alpha H + (\mu - m\alpha)G$. On remarking that $[G^2] = 0$, $[H] = p$ (here p is the genus of the ruled surface also), (8) gives

$$[C] = 1 + \mu(\alpha - 1) + \alpha(p - 1) - \frac{m\alpha(\alpha - 1)}{2}.$$

Finally, if

$$C' = \alpha' H + (\mu - m\alpha')G,$$

we have from (9)

$$[CC'] = \alpha\mu' + \mu\alpha' - m\alpha\alpha'.$$

Both these formulas are due to Segre (t). Observe that $G = 0$ requires $\alpha = \beta = 0$, which means that H and G satisfy no equivalence; and as they constitute a minimum base, we have $\rho = 2, \sigma = 1$.

13. *Existence Theorem for* $\{C\}$. In terms of the characters $[C]$, $[C^2]$, some noteworthy results concerning the dimension of $|C|$ or $\{C\}$ may be readily expressed. In order to avoid introducing new terms we only give a somewhat special proposition taken from the beautiful theory due mostly to Castelnuovo-Enriques, and Severi (*e*, 706):

Let $s = [C^2] - [C] + 1 - q$. *If* $s \geqq 0$, $\{C\}$ *exists and consists of* ∞^q *linear systems. Should the generic* C *be irreducible,* $|C|$ *is of dimension* $\geqq s$.

14. *Severi's Criterion.* In his first and most important paper on the base, appears this criterion for equivalence:

If A, B *are of the same order and if* $[A^2] = [AB] = [B^2]$ *then* $\lambda A = \lambda B, \lambda \neq 0$.

The proof outlined here constitutes a simplification of Severi's. Let $[B] \geqq [A]$, and set $H_t = H + t(A - B)$. From $[HB] = [HA]$, it follows

$$[H_t H] = [H^2] = m,$$

i.e., the order of H_t is fixed. It is then found, say by (8) and (9), that

$$[H_t^2] - [H_t] + 1 - q = m - p + t[B] - t[A] + 1 - q,$$

which approaches ∞ with t. Hence all systems $\{H_t\}$, with t above a certain limit, exist. As their curves are all of same order, the number of distinct systems among them is finite, so that, for example, $\{H_{t_1}\}$ and $\{H_{t_2}\}$, $t_1 \neq t_2$, coincide. Hence, we have at once

$$(t_1 - t_2)A = (t_1 - t_2)B.$$

From this criterion Severi concludes that in order that $\sum_{i=1}^{k} \lambda_i C_i = 0$, it is necessary and sufficient that the matrix

$$\left\| \begin{matrix} [C_1 H], & [C_2 H], & \cdots, & [C_k H] \\ [C_1^2], & [C_1 C_2], & \cdots, & \cdots \\ \cdot & \cdot & \cdot & \cdot \\ [C_1 C_k], & \cdots, & \cdots, & [C_k^2] \end{matrix} \right\| \tag{10}$$

be of rank $< k$.

Suppose for example that $C_1, C_2, \cdots, C_k$ are logarithmic curves of a certain integral of total differentials J with a

period α_i with respect to C_i. On any curve D not a C_i there is determined an Abelian integral with the $[DC_i]$ intersections of C_i and D for logarithmic points, the corresponding periods being α_i. It follows $\sum \alpha_i[DC_i] = 0$. As a matter of fact D may be replaced in this relation by any curve whatever, and therefore the matrix (10) is of rank $< k$. Hence the (C)'s are related by an equivalence. Conversely if they are so related J exists and is readily formed. This is essentially Severi's proof of his theorem stated in No. 11 (u, V).

15. *Determination of ρ and σ.* The actual determination of these elements for a given F is a difficult problem. A regular process may be given, but it has little practical value. However, various special methods have yielded the solution in all cases of interest. These cases fall mainly into two classes which we examine in turn.*

The first class consists of surfaces of sufficiently general nature contained in known algebraic varieties. The simple varieties (linear spaces or their loci, Abelian varieties) yield the most significant results. As early as 1882, Noether stated (p, II) the following important theorem: An arbitrary non-singular surface of order $m > 3$ situated in an S_3 (S_r = r-space) contains only curves that are complete intersections with other surfaces of S_3. It follows that for such a surface a plane section constitutes a minimum base, and therefore $\rho = \sigma = 1$. An incomplete proof of a geometric nature, of a similar theorem for surfaces that are complete intersections of $r - 2$ varieties in an S_r was given by Fano (j, II).†

* In ϵ (p. 730) will be found references pertaining to surfaces not discussed here.

† He makes an assumption leading to the impossibility for a double integral of the first kind to be without periods. The proof in n, IV, p. 358, is correct but for this exception noticed by Fano: the quartic surface intersection of two quadrics in S_4. The proof (loc. cit.) fails when the integer denoted there by n is negative, and a very simple discussion shows that when r exceeds three, this occurs only in the case just mentioned.

As beyond the scope of this report but noteworthy here must be mentioned extensions to algebraic varieties (Klein, m; Fano, j, I; Severi, u, VIII; Lefschetz, n, IV, p. 359).

A very general theorem from which the preceding may be derived was given by the writer by means of analysis situs (*n*, IV, p. 355). The following important special case will suffice as an indication. In a three-dimensional variety V_3 let $|F|$ be a linear system of surfaces, ∞^3 at least, of sufficiently general type (more or less analogous to the system of hyperplane sections). If the number of linearly independent double integrals of the second kind of a generic F exceeds that of V_3 (see *n*, IV), a base for the surfaces of V_3 intersects F into a similar one for its curves. V_3 and F have then equal numbers ρ, σ. (These numbers and the bases are defined for V_3 as for a surface.) Thus, to prove Noether's theorem, it is sufficient to show that F possesses double integrals of the first kind with periods not all zero, which can be done in this case (*n*, IV, p. 358; *n*, VII, Ch. 5). These will constitute integrals of the second kind not of type (5) (Picard, *q*, II, p. 365).

As S_3 possesses no double integrals of the second kind that are linearly independent, the theorem becomes applicable. A plane constitutes a minimum base for S_3 and its trace on F, that is a plane section, will be one for F. The surface being regular, Noether's theorem follows readily.

The second class of surfaces for which the determination of the bases has been carried out is composed of hyperelliptic surfaces and surfaces which are the image of pairs of points of two algebraic curves.

In a series of papers dating from 1893, G. Humbert made a searching investigation of hyperelliptic surfaces (*l*). Applying a result due to Appell, he showed in particular that the parametric equation of any curve traced on a hyperelliptic surface F is characterized by the vanishing of an intermediary function attached to the period matrix (entire function such that the addition of periods has merely the effect of multiplying it by a linear exponential) (*l*, first paper). From this and through an elegant analysis, Bagnera and de Franchis obtained the value of ρ and the bases even for hyperelliptic surfaces of very special type (*d*).

Surfaces images of point-pairs of two algebraic curves have been investigated by various authors (*e*, p. 730), the most important contribution being Severi's (*u*, I). In substance he shows that to every correspondence between the two curves corresponds a curve on F, and vice versa. Let λ be the number of singular correspondences between the two curves in the sense of Hurwitz. Severi's result gives readily $\rho = \lambda + 2$, $\sigma = 1$. However, he did not state this explicitly (loc. cit.) for his paper antedates by three years his first one on the base.

The surfaces considered in this article furnish ideal applications for the theorem (*b*) of § 8 (*n*, VII, Ch. 4). Results already known are obtained with great ease and elegance. Moreover the same method has been the basis for the extension to Abelian varieties (*n*, IV).

16. *Conclusion.* An outstanding question is the determination of ρ and the bases when only real curves are taken into consideration. So far as we know it has been solved only for rational surfaces (Comessati, *g*) and hyperelliptic surfaces (Lefschetz, *n*, V).

Algebraic varieties which we have excluded from this report are still somewhat *terra incognita*, although some important general theorems are known (*n*, IV). Perhaps we must await further information of a purely geometric nature before much progress can be expected.

References

a. J. W. Alexander, Rendiconti dei Lincei, (5), vol. 23 (1914), p. 55.

b. G. Albanese, Annali di Matematica, (3), vol. 4 (1915).

c. Appell et Goursat, *Théorie des Fonctions Algébriques et de leurs Intégrales*.

d. Bagnera-de Franchis, Palermo Rendiconti, vol. 30 (1910), p. 185.

e. Castelnuovo-Enriques, Encyklopädie, III C 6 b.

f. O. Chisini, Bologna Memorie, (7), vol. 8 (1920–21).

g. A. Comessatti, Mathematische Annalen, vol. 73 (1913); Annali di Matematica, (3), vol. 33 (1914).

h. F. Enriquês, Torino Atti, vol. 37 (1901), p. 25.

j. G. Fano, I. Torino Atti, vol. 39 (1903–04), p. 469.
II. Torino Atti, vol. 44 (1908–09), p. 15.

k. L. Godeaux, Bulletin des Sciences Mathématiques, vol. 39 (1915), p. 182.

l. G. Humbert, Journal de Mathématiques, (4), vol. 9 (1893); (5), vol. 5 (1899); (5), vol. 6 (1900); (5), vol. 7 (1901); (5), vol. 10 (1904).

m. F. Klein, Mathematische Annalen, vol. 22 (1887), p. 234.

n. S. Lefschetz, I. Annals of Mathematics, (2), vol. 17 (1916).
II. Annals of Mathematics, (2), vol. 21 (1920).
III. Rendiconti dei Lincei, (5), vol. 26 (1917), p. 228.
IV. Transactions of this Society, vol. 22 (1921).
V. Proceedings of the National Academy, vol. 5 (1916), p. 229.
VI. Comptes Rendus, vol. 176 (1923), p. 941.
VII. *L'Analysis Situs et la Géométrie Algébrique*, Paris, Gauthiers-Villars. (In press.)

o. B. Levi, Annali di Matematica, (2), vol. 26 (1897), p. 219.

p. M. Noether, I. Mathematische Annalen, vol. 2 (1870).
II. Berliner Abhandlungen, (1882) § 11.
III. Acta Matematica, vol. 8 (1886), p. 182.

q. Picard et Simart, *Traité des Fonctions Algébriques de Deux Variables*, Paris, Gauthier-Villars, two volumes, referred to respectively as *q*, I and *q*, II.

r. E. Picard, *Traité d'Analyse*, vol. 2, Ch. 14, second edition.

s. H. Poincaré, I. Journal de Mathématiques, (6), vol. 2 (1906).
II. Annales de L'École Normale, (2), vol. 27 (1910).
III. Sitzungsberichte der Berliner Mathematischen Gesellschaft, vol. 10 (1911).

t. C. Segre, Rendiconti dei Lincei, (4), vol. 3 (1887), p. 3; Mathematische Annalen, vol. 34 (1889), p. 2.

u. F. Severi, I. Torino Memorie, vol. 54 (1903).
II. Annali di Matematica, (3), vol. 12 (1905).
III. Palermo Rendiconti, vol. 21 (1906).
IV. Palermo Rendiconti, vol. 30 (1910).
V. Mathematische Annalen, vol. 62 (1905).
VI. Annales de L'École Normale, (2), vol. 25 (1908).
VII. Rendiconti dei Lincei, (5), vol. 15 (1916), p. 691.
VIII. Rendiconti dei Lincei, (5), vol. 17 (1908), p. 465.
IX. Rendiconti dei Lincei, (5), vol. 30 (1921).
X. Lombardo Rendiconti, (2), vol. 38 (1905), p. 859.
XI. Pascal, *Repertorium*, II_2, Ch. 33.

v. O. Veblen, *Cambridge Colloquium Lectures*, Part II: *Analysis Situs*, New York, 1922.

The University of Kansas

NOTE ON THE CONVERGENCE OF WEIGHTED TRIGONOMETRIC SERIES*

BY DUNHAM JACKSON

1. *Introduction.* Let $f(x)$ be a function continuous for all values of x, and of period 2π. Let $T_n(x)$ be a trigonometric sum of the nth order.† If $T_n(x)$ is determined, among all such sums, by the condition that the value of the integral

$$\int_0^{2\pi} [f(x) - T_n(x)]^2 dx$$

shall be a minimum, it becomes the partial sum of the Fourier series for $f(x)$. The problem can be generalized by taking, as the quantity to be reduced to a minimum, the integral

$$\int_0^{2\pi} \rho(x)[f(x) - T_n(x)]^2 dx, \tag{1}$$

where $\rho(x)$, indicating the *weight* to be attached to different values of the argument, is a function of x, likewise of period 2π, and positive for all values of x. There is a considerable body of literature bearing more or less directly on the generalized problem. This literature owes its inspiration largely to the researches of Tchebychef;‡ particular mention should also be made of a classical memoir by Gram.§

The purpose of the following paragraphs is to discuss the convergence of $T_n(x)$ toward the value $f(x)$, as n becomes infinite. The method is one which I have used recently in connection with the corresponding problem in which the weight is constantly equal to unity, and the square of the error is replaced by a power with a different exponent. The

* Presented to the Society, December 30, 1920.

† The words "of the nth order" will be understood throughout to mean "of the nth order at most."

‡ Cf., e.g., H. Burkhardt, *Entwicklungen nach oscillirenden Functionen und Integration der Differentialgleichungen der mathematischen Physik*, JAHRESBERICHT DER VEREINIGUNG, vol. 10, Heft 2 (1908), pp. 823 ff.

§ J. P. Gram, *Ueber die Entwickelung reeller Functionen in Reihen mittelst der Methode der kleinsten Quadrate*, JOURNAL FÜR MATHEMATIK, vol. 94 (1883), pp. 41–73.

question of convergence is treated by Gram, in the paper cited, but scarcely in a manner to meet the requirements of modern analysis.* More recently it has come within the range of a number of investigations, including a series of papers by Stekloff, in the BULLETIN DE L'ACADÉMIE DES SCIENCES, PETROGRAD, and elsewhere, with which I am only very imperfectly acquainted; a paper by J. Chokhate,† which I have seen in manuscript; and a series of papers by Szegö.‡ Up to the present time, I have not seen any treatment covering precisely the results that are presented below. If it should appear nevertheless that such a treatment exists, the novelty of this paper would consist in the method employed, and in the applicability of the method to the case in which the exponent 2 in (1) is replaced by an arbitrary m, as suggested in the concluding paragraph.

2. *The Convergence Theorem.* The conclusion to be established is as follows:§

Let $\omega(\delta)$ be the maximum of $|f(x') - f(x'')|$ for $|x' - x''| \leqq \delta$. Let $\rho(x)$ be continuous and positive for all values of x; or, if not continuous, let it be measurable, and always included between two fixed positive bounds.|| Then we may state the theorem:

---

* Cf. Burkhardt, loc. cit., pp. 848–854.

† See also J. Chokhate, *Sur quelques propriétés des polynomes de Tchébicheff*, COMPTES RENDUS, vol. 166 (1918), pp. 28–31.

‡ G. Szegö, *Über die Entwickelung einer analytischen Funktion nach den Polynomen eines Orthogonalsystems*, MATHEMATISCHE ANNALEN, vol. 82 (1921), pp. 188–212; *Über die Entwicklung einer willkürlichen Funktion nach den Polynomen eines Orthogonalsystems*, MATHEMATISCHE ZEITSCHRIFT, vol. 12 (1922), pp. 61–94; *Über den asymptotischen Ausdruck von Polynomen, die durch eine Orthogonalitätseigenschaft definiert sind*, MATHEMATISCHE ANNALEN, vol. 86 (1922), pp. 114–139; and other papers referred to in footnotes attached to the above.

§ The proof of the existence of a unique solution for the minimum problem is based so directly on similar proofs already given that it will not be taken up in detail here; cf. D. Jackson, *On functions of closest approximation*, TRANSACTIONS OF THIS SOCIETY, vol. 22 (1921), pp. 117–128.

|| It would of course make no difference if this condition were violated at points of a set of measure zero, since the value of the integral (1), and consequently the determination of $T_n(x)$, would not be affected.

The sum $T_n(x)$ *will converge uniformly to the value* $f(x)$ *for* $n \to \infty$ *provided that**

$$\lim_{\delta \to 0} \omega(\delta)/\sqrt{\delta} = 0.$$

As already stated, the proof is similar to one given recently in another connection.† In the first place, if $f(x)$ and $\varphi(x)$ are two functions whose difference is a trigonometric sum $t_n(x)$ of order n:

$$f(x) = \varphi(x) + t_n(x),$$

and if $T_n(x)$ and $\tau_n(x)$ are two sums, likewise of order n, such that

$$T_n(x) = \tau_n(x) + t_n(x),$$

the value of the integral (1) formed with $f(x)$ and $T_n(x)$ is the same as the value of the corresponding integral formed with $\varphi(x)$ and $\tau_n(x)$, and both integrals will reach their minimum values simultaneously. That is, if $T_n(x)$ and $\tau_n(x)$ represent the best approximating functions for $f(x)$ and $\varphi(x)$, respectively, as judged by the value of the integral (1), the errors $f(x) - T_n(x)$ and $\varphi(x) - \tau_n(x)$ will be identical.

By a general theorem on the approximate representation of continuous functions,‡ there will exist sums $t_n(x)$, of all orders $n > 0$, such that the difference between $f(x)$ and $t_n(x)$ never exceeds a constant multiple of $\omega(2\pi/n)$. In formulas, let

$$\varphi_n(x) = f(x) - t_n(x),$$

and let ϵ_n be the maximum of $|\varphi_n(x)|$; then

$$\epsilon_n \leqq c\omega(2\pi/n),$$

where c is independent of n. In particular, if $\omega(\delta)$ satisfies

* There is no reason to suppose that the particular infinitesimal $\sqrt{\delta}$ has any essential significance for the problem; its occurrence is in all probability due merely to the limitations of the method.

† D. Jackson, *On the convergence of certain trigonometric and polynomial approximations*, TRANSACTIONS OF THIS SOCIETY, vol. 22 (1921), pp. 158–166.

‡ Cf., e.g., D. Jackson, *On the approximate representation of an indefinite integral and the degree of convergence of related Fourier's series*, TRANSACTIONS OF THIS SOCIETY, vol. 14 (1913), pp. 343–364; p. 350.

the hypothesis of the theorem,

$$\lim_{n\to\infty} \epsilon \sqrt{n} = 0. \tag{2}$$

Let $\tau_n(x)$ be the trigonometric sum of order n which gives the best approximation to $\varphi_n(x)$, as determined by the integral corresponding to (1); let

$$\gamma_n = \int_0^{2\pi} \rho(x)[\varphi_n(x) - \tau_n(x)]^2 dx;$$

and let $\mu_n = |\tau_n(x_0)|$ be the maximum of $|\tau_n(x)|$. Let it be assumed that

$$0 < v \leqq \rho(x) \leqq V,$$

the numbers v and V being constants; and let it be assumed temporarily that $\mu_n \geqq 4\epsilon_n$.

By Bernstein's theorem,* since

$$|\tau_n(x)| \leqq \mu_n,$$

it follows that

$$|\tau_n'(x)| \leqq n\mu_n$$

for all values of x. In particular, for values of x in the interval

$$|x - x_0| \leqq \frac{1}{2n},$$

it can be inferred that

$$|\tau_n(x) - \tau_n(x_0)| \leqq \frac{\mu_n}{2},$$

and

$$|\tau_n(x)| \geqq \frac{\mu_n}{2}.$$

Since

$$|\varphi_n(x)| \leqq \epsilon_n \leqq \mu_n/4,$$

it follows further that

$$|\varphi_n(x) - \tau_n(x)| \geqq \frac{\mu_n}{4}$$

throughout the interval specified, and, as the length of the interval is $1/n$, and $\rho(x) \geqq v$,

$$\gamma_n \geqq \frac{v}{n}\left(\frac{\mu_n}{4}\right)^2.$$

* See, e.g., de la Vallée Poussin, *Leçons sur l'Approximation des Fonctions d'une Variable Réelle*, Paris, 1919, pp. 39–42.

On the other hand, by the minimum property of $\tau_n(x)$, the value of γ_n is less than that which would be obtained if $\tau_n(x)$ were replaced by any other trigonometric sum of order n; in particular, by comparison with the integral which is obtained if 0 is substituted for $\tau_n(x)$,

$$\gamma_n \leqq 2\pi V \epsilon_n^2.$$

Hence

$$\frac{v}{n}\left(\frac{\mu_n}{4}\right)^2 \leqq 2\pi V \epsilon_n^2,$$

$$\mu_n \leqq 4\sqrt{\frac{2\pi V}{v}}\ \epsilon_n \sqrt{n}.$$

This relation, derived on the hypothesis that $\mu_n \geqq 4\epsilon_n$, clearly holds in the contrary case also, since $V \geqq v$ and $n \geqq 1$.

In any case, then, since $|\varphi_n| \leqq \epsilon_n$ and $|\tau_n| \leqq \mu_n$,

$$|\varphi_n(x) - \tau_n(x)| \leqq \epsilon_n + 4\sqrt{\frac{2\pi V}{v}}\ \epsilon_n \sqrt{n} \leqq k\epsilon_n \sqrt{n},$$

where k is independent of n. But it has been pointed out already that $\varphi_n(x) - \tau_n(x)$ is the same as $f(x) - T_n(x)$, where $T_n(x)$ is the sum giving the best approximation to $f(x)$, as determined by the integral (1); hence

$$|f(x) - T_n(x)| \leqq k\epsilon_n \sqrt{n}.$$

This relation, combined with (2), establishes the truth of the theorem.

With the same method of treatment, the problem can be varied by using a general power of the absolute value of the error, instead of the square, together with a weight-function $\rho(x)$; and the method is applicable also to problems of polynomial approximation. For treatment in detail, however, the case discussed above may be regarded as sufficiently illustrative.

THE UNIVERSITY OF MINNESOTA

PROOF OF A FORMULA FOR AN AREA*

BY H. E. BRAY

1. *Introduction.* The formula

$$A = \int\int_R \frac{\partial(x, y)}{\partial(u, v)} dudv,$$

representing the area of the image of a (u, v) rectangle R in the (x, y) plane, can be proved to hold in the case where the transformation $x = x(u, v)$, $y = y(u, v)$ is of a very general kind.† The purpose of this paper is to extend and prove the formula by means of an approximating function of a very simple kind. The main properties of this function are given in § 3 and used in the proof of the formula, § 4.

2. *Definitions.* The approximating function for the summable function $f(x, y)$ is given by the formula

$$f^{(\mu)}(x, y) = \frac{1}{\sigma_\mu} \int\int_{\sigma_\mu} f(x + \xi, y + \eta) d\xi d\eta$$

where σ_μ represents the square region $[-\mu \leqq \xi \leqq \mu, -\mu \leqq \eta \leqq \mu]$, also the area of that square.‡ For convenience $f(x, y)$ is regarded as summable (Lebesgue) in the fundamental region S $[0 \leqq x \leqq 1, 0 \leqq y \leqq 1]$ and the properties of $f^{(\mu)}$ are considered with reference to a rectangle R $[a \leqq x \leqq b, c \leqq y \leqq d]$ inside S, μ being less than a, c, $1 - b$, $1 - d$.

The formula to be proved involves *generalized derivatives* and *potential functions*. These are defined as follows:

DEFINITION (i). If α is a given direction and α' the direction 90° in advance of α, the quantity

$$D_\alpha f(x, y) = \lim_{\sigma \to 0} \frac{1}{\sigma} \int_s f(x, y) d\alpha',$$

if it exists, is called the *generalized derivative of f in the direction* α. It is understood that the integral is taken in the positive

* Presented to the Society, December 27, 1922.

† W. H. Young, PROCEEDINGS OF THE LONDON SOCIETY, (2), vol. 18, p. 339.

‡ It is convenient sometimes to use as σ_μ the interior of the circle $\xi^2 + \eta^2 = \mu^2$. The properties of $f^{(\mu)}$ are essentially the same in this case.

sense around the contour s, of a square σ, having its center at (x, y) and its sides parallel to the coordinate axes.§

DEFINITION (ii). Let $\varphi(x, y)$ be a vector point function whose component φ_α in every fixed direction, α, is summable superficially, and $u(x, y)$ a scalar point function, summable superficially, and such that $\int_s u(x, y)d\alpha$ may be defined for every given direction α. Then if φ and u are so related that $\int\int_\sigma \varphi_\alpha d\sigma = \int_s u d\alpha'$ for every rectangle σ* and for every fixed direction α, u is said to be a *potential function for the vector* φ.

3. *Properties of the Function* $f^{(\mu)}$.†

(1) If $f(x, y)$ is summable superficially $f^\mu(x, y)$ is a continuous function of (x, y), absolutely continuous in x for every y and in y for every x. At nearly every point of any line $y = \text{const.}$ $\partial f^{(\mu)}/\partial x$ exists:

$$\frac{\partial}{\partial x} f^{(\mu)} = \frac{1}{\sigma_\mu} \int_{s_\mu} f(x + \xi, y + \eta) d\eta,$$

the integral being evaluated about the contour s_μ of σ_μ.‡

(2) If $f(x, y)$ is absolutely continuous in x for every y and if $\partial f/\partial x$ is summable superficially, then, if y is given, for nearly every value of x,

$$\frac{\partial}{\partial x} f^{(\mu)} = \left(\frac{\partial f}{\partial x}\right)^{(\mu)}.$$

(3) If $f(x, y)$ is a potential function for its generalized derivatives, then, if y is given, for nearly every x

$$\frac{\partial}{\partial x} f^{(\mu)} = (D_x f)^\mu.$$

(4) If $f(x, y)$, besides satisfying the conditions of (3), is also

§ G. C. Evans, *Fundamental problems of potential theory*, RICE INSTITUTE PAMPHLETS, vol. 7, No. 4 (Oct., 1920). In the original definition Professor Evans considers a general class Γ of curves. A summary of this paper is given in PROCEEDINGS OF THE NATIONAL ACADEMY, vol. 7 (1921).

* Here, again, the definition as given by Professor Evans refers to the general class Γ of curves.

† The proofs are simple and are omitted, in most cases, for brevity.

‡ $f^{(\mu)}$ can be generalized when higher derivatives are required; e.g.,

$$f^\mu(x, y) = \frac{1}{\sigma_\mu^2} \int\int_{\sigma_\mu} d\xi' d\eta' \int\int_{\sigma_\mu} f(x + \xi + \xi', y + \eta + \eta') d\xi d\eta$$

possesses second derivatives nearly everywhere.

continuous in (x, y), then

$$\frac{\partial}{\partial x} f^{(\mu)} = (D_x f)^{(\mu)},$$

at every point (x, y).

(5) If $f(x, y)$ is a potential function for its generalized derivatives, then $D_x(f^{(\mu)})$ exists and is equal to $(D_x f)^{(\mu)}$.

(6) If $f(x, y)$ is summable superficially, the function $f(x + \xi, y + \eta)$ is measurable and summable in the three-dimensional region $[a \leqq s \leqq b, -\mu \leqq \xi \leqq \mu, -\mu \leqq \eta \leqq \mu]$, where s is the arc of a rectifiable curve upon which the point (x, y) lies, and

$$\int_{s_1}^{s_2} f^{(\mu)}(x, y)ds = \frac{1}{\sigma_\mu}\int_{s_1}^{s_2} ds \int\int_{\sigma_\mu} f(x + \xi, y + \eta)d\xi d\eta$$
$$= \frac{1}{\sigma_\mu}\int\int_{\sigma_\mu} d\xi d\eta \int_{s_1}^{s_2} f(x + \xi, y + \eta)ds.$$

(7) Similarly $f(x + \xi, y + \eta)$, regarded as a function of four variables (x, y, ξ, η), is summable in the region

$$[a \leqq x \leqq b, \quad c \leqq y \leqq d, \quad -\mu \leqq \xi \leqq \mu, \quad -\mu \leqq \eta \leqq \mu].$$

Consequently if σ is the rectangle $[a \leqq x \leqq b, c \leqq y \leqq d]$

$$\int\int_\sigma f^{(\mu)}(x, y)dxdy = \frac{1}{\sigma_\mu}\int\int_\sigma dxdy \int\int_{\sigma_\mu} f(x + \xi, y + \eta)d\xi d\eta$$
$$= \frac{1}{\sigma_\mu}\int\int_{\sigma_\mu} d\xi d\eta \int\int_\sigma f(x + \xi, y + \eta)dxdy.$$

(8) Since $\int\int_\sigma f(x + \xi, y + \eta)dxdy$ is a continuous function of (ξ, η),

$$\lim_{\mu \to 0} \int\int_\sigma f^{(\mu)}(x, y)dxdy$$
$$= \lim_{\sigma_\mu \to 0} \frac{1}{\sigma_\mu}\int\int_{\sigma_\mu} d\xi d\eta \int\int_\sigma f(x + \xi, y + \eta)dxdy$$
$$= \int\int_\sigma f(x, y)dxdy.$$

(9) If $f(x, y)$ and $g(x, y)$ are two functions, summable with their squares, then

$$\lim_{\mu \to 0} \int\int_\sigma f^{(\mu)} g^{(\mu)} dxdy = \int\int_\sigma fgdxdy.$$

In fact since f^2 and g^2 are summable, it can be shown by

Schwarz's inequality that the absolute continuity of the integral $\int\int |f^{(\mu)}g^{(\mu)}|dxdy$ is uniform * for all μ, and this is sufficient to prove the property.

(10) If $f(x, y)$ is continuous in (x, y), $f^{(\mu)}$ approaches f uniformly as μ approaches 0. If f is merely summable, $f^{(\mu)}$ approaches f at nearly every point (x, y).

(11) If $f(x, y)$ is a function of limited variation in x, uniformly for every y, then $f^{(\mu)}$ is a function of limited variation in x, uniformly for all values of y and μ. In fact, if

$$\sum_i |f(x_{i+1}, y) - f(x_i, y)| \leqq T_x,$$

a constant, then

$$\sum_i |f^{(\mu)}(x_{i+1}, y) - f^{(\mu)}(x_i, y)|$$

$$\leqq \frac{1}{\sigma_\mu}\int\int_{\sigma_\mu}\sum_i |f(x_{i+1}+\xi, y+\eta) - f(x_i+\xi, y+\eta)|\,d\xi d\eta \leqq T_x.\dagger$$

4. *Proof of the Formula.* It has been shown by W. H. Young ‡ that, given the continuous transformation $x = x(u, v)$, $y = y(u, v)$, the area of the image of the rectangle R $[a \leqq u \leqq b, c \leqq v \leqq d]$ is given by the formula

$$A = \int\int_R \frac{\partial(x, y)}{\partial(u, v)}\,dudv.$$

He assumes the absolute continuity of x and y as functions of u alone and of v alone, the absolute continuity with regard to u being uniform for a dense set of values of v in the interval $c \leqq v \leqq d$. In the theorem which follows absolute continuity of x and y is not assumed; instead it is assumed that x and y are potential functions for their generalized derivatives.

THEOREM. *If $x(u, v)$, $y(u, v)$ are continuous functions of (u, v), if each is a potential function for its generalized derivatives, the latter being summable with their squares, and if y (or x) is a function of uniformly limited variation in u for every v, and in v for every u, then the area of the image of the rectangle*

* C. de la Vallée-Poussin, TRANSACTIONS OF THIS SOCIETY, vol. 16 (1915), pp. 445 et seq.

† It is sufficient to assume that the total variation with regard to x is less than some function of y, $c(y)$, summable in y and such that

$$\int_y^{y'} c(y)dy < M(y' - y).$$

‡ W. H. Young, loc. cit.

$R[a \leqq u \leqq b, c \leqq v \leqq d]$ is given by the formula

$$A = \int\int_R (D_u x D_v y - D_v x D_u y) du dv.$$

It is shown by Young * that the area A is equal to

$$\tfrac{1}{2} \int_{C_R} x dy - y dx,$$

where the integrals are Stieltjes integrals evaluated about the contour C_R of R. By a known property of the Stieltjes integral,† namely,

$$\int_a^b x dy = xy \Big]_a^b - \int_a^b y dx,$$

it follows, under our hypothesis, that

$$A = \int_{C_R} x dy.$$

Instead of this expression we consider the approximation

$$A_\mu = \int_{C_R} x^{(\mu)} dy^{(\mu)} = \int_{C_R} x^{(\mu)} \frac{\partial}{\partial u} y^{(\mu)} du + x^{(\mu)} \frac{\partial}{\partial v} y^{(\mu)} dv.$$

The change to the last form is justified by the fact that $x^{(\mu)}$, $y^{(\mu)}$ are continuous with their first derivatives. By property (4) this can be written in the form

$$\int_{C_R} x^{(\mu)} (D_u y)^{(\mu)} du + x^{(\mu)} (D_v y)^{(\mu)} dv,$$

and since the integrands are absolutely continuous

$$A_\mu = \int_a^b du \int_c^d - \frac{\partial}{\partial v} [x^{(\mu)} (D_u y)^{(\mu)}] dv + \int_c^d dv \int_a^b \frac{\partial}{\partial u} [x^{(\mu)} (D_v y)^{(\mu)}] du.$$

The quantity $\partial (D_v y)^{(\mu)} / \partial u$ is summable superficially, being equal to the contour integral of $D_v y$ about a square (properties (1) and (6)). Moreover for every rectangle $r[a' \leqq u \leqq b', c' \leqq v \leqq d']$,

$$\int\int_r \left[\frac{\partial}{\partial u} (D_v y)^{(\mu)} \right] du dv = \int\int_r \left[\frac{\partial}{\partial v} (D_u y)^{(\mu)} \right] du dv,$$

* W. H. Young, loc. cit.

† H. E. Bray, Annals of Mathematics, (2), vol. 20 (1918-19), p. 185.

Since

$$\int\int_r \left[\frac{\partial}{\partial u}(D_v y)^{(\mu)}\right] du dv = \left[\int_{c'}^{d'} (D_v y)^{\mu} dv\right]_{u=a'}^{u=b'}$$
$$= \left[\int_{c'}^{d'} \frac{\partial}{\partial v} y^{(\mu)} dv\right]_{u=a'}^{u=b'}$$
$$= y^{(\mu)}(b', d') - y^{(\mu)}(b', c') - y^{(\mu)}(a', d') + y^{(\mu)}(a', c').$$

Consequently, x being continuous, for every rectangle r,

$$\int\int_r x^{(\mu)} \frac{\partial}{\partial u}(D_v y)^{(\mu)} du dv = \int\int_r x^{(\mu)} \frac{\partial}{\partial v}(D_u y)^{(\mu)} du dv.$$

We can now express the quantity A_μ as a double integral and on cancelling the terms just mentioned we obtain

$$A_\mu = \int\int_R \left[\frac{\partial}{\partial u} x^{(\mu)}(D_v y)^{(\mu)} - \frac{\partial}{\partial v} x^{(\mu)}(D_u y)^{(\mu)}\right] du dv,$$

and since

$$\frac{\partial}{\partial u} x^{(\mu)} = (D_u x)^{(\mu)}, \qquad \frac{\partial}{\partial v} x^{(\mu)} = (D_v x)^{(\mu)},$$

by property (4),

$$A_\mu = \int\int_R [(D_u x)^{(\mu)}(D_v y)^{(\mu)} - (D_v x)^{(\mu)}(D_u y)^{(\mu)}] du dv.$$

Now let μ approach zero. Since the generalized derivatives are, by hypothesis, summable with their squares, by (9),

$$\lim_{\mu \to 0} A_\mu = \int\int_R [D_u x D_v y - D_v x D_u y] du dv.$$

To show that $\lim_{\mu \to 0} A_\mu = \int_{C_R} x dy$, consider the following quantity, in which the integrals are taken in regard to u for a fixed value of v:

$$\left| \int_{u=a}^{u=b} x^{(\mu)} dy^{(\mu)} - \int_{u=a}^{u=b} x dy \right|$$
$$\leqq \left| \int x^{(\mu)} dy^{(\mu)} - \int x dy^{(\mu)} \right| + \left| \int x dy^{(\mu)} - \int x dy \right|$$
$$= \left| \int (x^{\mu} - x) dy^{(\mu)} \right| + \left| \int x dy^{(\mu)} - \int x dy \right|.$$

The first term is not greater than $\max|x^{(\mu)} - x|$ times the total variation of $y^{(\mu)}$; and since the latter is uniformly limited for every μ and $x^{(\mu)}$ approaches x uniformly, this term approaches zero with μ. The second term approaches zero

because $y^{(\mu)}$ approaches y and is of uniformly limited variation in u for every μ,* by (11). The quantity $\int_{u=a}^{u=b} x^{(\mu)}dy^{(\mu)}$ is typical of those which constitute A_μ, consequently

$$\lim_{\mu \to 0} A_\mu = \int_{C_R} xdy = A.$$

We have thus proved that

$$A = \int\int [D_u x D_v y - D_v x D_u y] du dv.$$

It has been shown by Evans † that if a continuous function is a potential function for its generalized derivatives, then its ordinary derivatives exist nearly everywhere and are equal to the corresponding generalized derivatives nearly everywhere. The formula which we have proved remains true, consequently, if ordinary derivatives are substituted for the generalized derivatives. On the other hand, if $\partial x/\partial u$, $\partial x/\partial v$ are given, a vector φ is defined whose components in the u and v directions are $\partial x/\partial u$, $\partial x/\partial v$, respectively, and

$$\varphi_\alpha = \frac{\partial x}{\partial u} \cos(u, \alpha) + \frac{\partial x}{\partial v} \sin(u, \alpha).$$

If $\partial x/\partial u$, $\partial x/\partial v$ are summable the same is true for any component φ_α and if

$$\int\int_\sigma \frac{\partial x}{\partial u} du dv = \int_{C_\sigma} x dv, \qquad \int\int_\sigma \frac{\partial x}{\partial v} du dv = -\int_{C_\sigma} x du,$$

x will be a potential function for the vector φ which is called its gradient vector.‡ Hence we have the following theorem.

THEOREM. *If $x(u, v)$, $y(u, v)$ are continuous functions of (u, v), if each is a potential function for its gradient vector, if y (or x) is of uniformly limited variation in u for every v and in v for every u, and if $\partial x/\partial u$, $\partial x/\partial v$, $\partial y/\partial u$, $\partial y/\partial v$ are summable with their squares, then*

$$A = \int\int_R \frac{\partial(x, y)}{\partial(u, v)} du dv.$$

RICE INSTITUTE

* H. E. Bray, loc. cit.

† G. C. Evans, RICE INSTITUTE PAMPHLETS, vol. 7, No. 4, pp. 97–99.

‡ G. C. Evans, loc. cit.

MATHEMATICS FOR THE LAYMAN

Mathematical Philosophy. A Study of Fate and Freedom. By Cassius J. Keyser. New York, E. P. Dutton and Co., 1922. 14 + 466 pp.

An author who attempts to give a popular exposition of any technical doctrine, be it in science, in art, or in philosophy, who (to use Professor Keyser's own words) aims to make a "contribution to the democratization of science and scientific criticism," faces a peculiarly difficult dilemma. He must beguile the layman's interest without being superficial; he must achieve adequate depth without being dull. A work of popularization, moreover, to fulfill its highest mission must be not merely descriptive or informative, but also interpretative. No one would probably deny the crying need of our age for popularizing books in all fields of technical inquiry satisfying these demands. And yet how meager the supply!

Professor Keyser has succeeded in meeting these conflicting demands in a remarkable way. His lectures were prepared primarily for students of philosophy; his book is addressed, however, to "educated laymen" in general. The appeal of its interest is very wide. It aims to discuss "the nature of mathematics, its significance in thought, and its bearings on human life." A large program, carried through with notable success! This does not mean, of course, that the last word has been said, that there is not room for difference of opinion, that many important fields of inquiry have not been left untouched. It does mean, however, that Professor Keyser has made a very significant contribution to the solution of the problem he set himself and that he has set up a standard of excellence which books of similar purpose and scope will find it difficult to meet.

The introductory lecture contains a discussion of the fundamental aims of education which is one of the finest things in the book. Many of our influential educators would confer a boon upon their country were they to repair to a mountain top, far away from the tangled undergrowth of short-sighted policy, and there read and ponder this discussion and let their souls commune with the educational ideal there presented. Such an educator might return to the uncharted wilderness of practical problems with a new vision and a more reliable compass.

The next eight lectures deal with various aspects of postulate systems. The point of view is that of Russell and Whitehead and much emphasis is laid on the alleged identity of pure mathematics and symbolic logic. Personally I would question the desirability of arbitrarily restricting so well established a term as "pure mathematics" to the meaning implied in such identification. Why not call the class of doctrinal functions "abstract mathematics" or "formal mathematics," or, if one really believes in the alleged identity, "formal logic"? Quite aside from this verbal problem, however, it seems to me that there is a real distinction between the problem of formal logic and the investigation of the various doctrinal functions—

a distinction at least of motive which I think is vital and should not be obscured. The motive of the logician is logical consistency or the validity of mental processes as such, the motive of the mathematician is the conclusions reached by such processes, their validity being by him assumed. Professor Keyser himself is not altogether consistent in his treatment of this subject. We may well argue that pure mathematics and formal logic are "organically related as are the roots of a tree and its branches," without thereby asserting that the roots and the branches are "the same thing." Furthermore, the definition of mathematics as a class of doctrinal functions—especially in the older form of Russell as "The class of all propositions of the form P implies Q"—leaves out, it seems to me, an essential characteristic of mathematics, viz., its structure. Mathematics is not a mere class—it is at least an ordered class of some sort.

A first example to illustrate the possibility of difference of opinion. Some others may appear later in this review. Let me say immediately that these opportunities for questioning constitute one of the most valuable features of the book. It is stimulating and thought provoking to a high degree.

I have used the phrase "doctrinal function." It is a very happy one and constitutes, I venture to say, a permanent addition to the vocabulary of our science. In proposing it Professor Keyser has done more than merely give an appropriate label. The concept which it describes was doubtless latent in the minds of many before its introduction. But by giving it a name he has precipitated the concept in precise form—and the concept is a valuable one.

Throughout these first nine lectures he develops also the thesis implied by his subtitle "A Study of Fate and Freedom." He calls attention at the very beginning to the eternal quality of ideas and their interrelations, to the immutable laws of thought to which the intellect is subject. "The world of ideas is the empire of Fate" (p. 5). After having developed the concept of a postulate system he returns to his thesis as follows (p. 136): "When once the principles, or postulates, are chosen, the die is cast—all else follows with a necessity, a compulsion, an inevitability that are absolute —we are at once subject to a destiny of consequences which no man nor any hero nor Zeus nor Yahweh nor any god can halt, annul or circumvent. Mathematics is in a word the study of Fate. Let me hasten to say that the Fate is not physical, it is spiritual. . . . The Fate is logical Fate. Is it a tyrant? And the intellect, then, a slave? . . . Where then is the intellect's freedom? What do you love? Poetry? Painting? . . . Music? The muses are *their* fates. If you love them you are free. Logic is the muse of *thought*. When I violate it, I am erratic; if I hate it, I am licentious or dissolute; if I love it, I am free—the highest blessing the austerest muse can give."

Here and throughout the book the author makes out a strong case for Fate—but it seems to me he leaves the case for Freedom unnecessarily weak. So much could have been said—and said beautifully, eloquently, by this particular author. I am sorry he did not say it. Is it true that the die is cast when the postulates have been chosen? What of the wealth

of definitions that are possible under a given set of postulates? What of the infinite variety of combinations of the primitive elements which the creative imagination can construct? Royce has called it the "eternal fairyland of mathematical construction." The author would perhaps challenge this point of view—would claim that the imagination does not *create* these combinations, that they are present from the start, and that the mind merely *discovers* them. Some of the author's remarks on the early pages of the book seem to imply this attitude. It seems to me, however, that such an attitude, while perhaps philosophically tenable, is logically unnecessary and esthetically unsatisfying. Did Beethoven discover his Fifth Symphony, or did he create it? Did Georg Cantor discover his Mengenlehre or did he create it? Was the group concept, were the manifold, beautiful developments in the recent theory of functions of real variables discovered or created? No dogmatic answer is perhaps possible —in that the answer depends on the philosophical temperament of the one who replies. But it seems to me at any rate far more satisfying, and equally true, to regard such conceptions as the creations of an artistic imagination. The recent work on the foundations seems to me to have contributed not a little to a better understanding of mathematics as a fine art, like music or painting. That Professor Keyser is fully sympathetic toward such a conception no one can doubt—it is, therefore, a bit puzzling that he does not make more of it in this book. The volume is in itself a fine example of the play of imagination, but it has little to say of imagination as such and the rôle it plays in mathematics.

After his treatment of postulate systems follow lectures on Transformation, Invariance, Group Concept, Limits, Infinity, Hyperspaces, Non-euclidean Geometries, Psychology, Korzybski's Concept of Man, and Science and Engineering. These lectures are so good that they deserve a critical and discriminating review. However, space forbids; and Professor Keyser has assured me that he would prefer a review of general character, as more in keeping with the purpose of the book. Furthermore, an adequate appreciation and criticism must be a composite from many sources. Let philosophers quarrel with the author, if they must, concerning his philosophical tenets; let scientists object, if they will, to the proposition that "mathematics is the prototype which every branch of science approximates in proportion as its basal assumptions and concepts become clearly defined" (but let them not venture into the argument until they have an adequate conception of the broad sense in which the term mathematics is used); let literary critics challenge, if they be so disposed, the author's analysis of the function and art of criticism; and let sociologists and engineers evaluate the author's conception of their proper function in accordance with Korzybski's idea of the nature of our human kind.

Enough has already been said to indicate my conviction that Professor Keyser has made a valuable contribution to the literature of popularization in the highest and best sense of this much abused term. His book accurately presents the more fundamental conceptions which form the basis of modern mathematics. Of this I am confident, even though I have read the book attentively (and pleasurably) rather than critically. I found

little to criticize in the statement of fact—perhaps no thing that would not lay me open, in view of the book's purpose, to the charge of quibbling.* He has presented these conceptions with a wealth of illustration, and in a style that is always pleasing and often of rare beauty and power. He has developed many and often surprising connections and analogies with apparently remote fields of inquiry. I venture to say that no one, be he professional mathematician or educated layman, can read this book without feeling its stimulating and thought-provoking character, provided only he be philosophically minded. A man not interested in meditating on the general aspect of things would perhaps find the book dull. But what a lot of the joy of life such a man must miss.

J. W. YOUNG

TWO TRANSLATIONS OF ARCHIMEDES

Les Œuvres Complètes d'Archimède. Traduites du Grec en Français avec une introduction et des notes. By Paul Ver Eecke. Paris and Brussels, Desclée, de Brouwer et Cie., 1921. lx + 553 pp.

Kugel und Zylinder von Archimedes. Uebersetzt und mit Anmerkungen versehen. By Arthur Czwalina-Allenstein. No. 202 of Ostwald's Klassiker der exakten Wissenschaften. Leipzig, Akademische Verlagsgesellschaft, 1922. 80 pp.

In considering these two recent evidences of Belgian and German scholarship it may naturally be asked why a new edition of the complete works of Archimedes, or even of a single treatise, should be thought worthy of publication at this time, particularly in view of the fact that we already have the monumental edition by Heiberg, with its recent revision; the

* In the interest of removing minor blemishes in a future edition, attention may be called to the following: In the group definition of the geometry of shape on p. 218 reference should be to "each and all the transformations of the similitude group" and *no others*; on p. 267, line 7 from the bottom, after the word "field" the restriction ($n \neq n'$) should be added; on p. 329 the statement that a plane of circles is "as rich in circles as in point-triads, as rich in circles as ordinary space in points" is erroneous unless the point-triads be restricted to those formed from points of a line, and is open to misunderstanding, since it leads rather easily to the erroneous idea that dimensionality is a function of the cardinal number of a class rather than of the arrangement of its elements. The extended treatment of the concept of limit seems to me unnecessarily involved and difficult; this portion of the book is hard reading even for one familiar with the concept.

Very few typographical errors were noticed. These occur on p. 136, line 5 from the bottom; on p. 175, line 4 from the bottom; on p. 243 lines 9 and 10; on p. 271, line 11; and on p. 377, line 12.

English edition, with a biographical study, by Heath; the French edition of Peyrard; the German translation of Nizze, and numerous other translations, more or less complete, by such earlier writers as Tartaglia, Venatorius (Thomas Jaeger), Commandinus, Revault (Rivaltus), Maurolycus, and Torelli.

There are various legitimate answers to the question, depending upon the point of view implied. In the first place, the older translators were usually limited to a single manuscript of each work, and that not a very reliable one; textual criticism, in the modern sense, was unknown. In the second place, Heiberg himself, the greatest student of Greek mathematical manuscripts who has undertaken to give a definitive text of Archimedes, based upon the most trustworthy codices, found that the discoveries of Papadopoulos Kerameus, in 1899, necessitated an extended revision of his own edition, and this revision was published in 1913–1915. It follows that all of the earlier translations must now be looked upon as obsolete, if we wish for correct versions. The significance of this statement may be appreciated by comparing the text of Peyrard (Paris, 1807), for example, with that of Heiberg or with the ones under review. Peyrard gives as the first proposition of the *Sphere and Cylinder* a statement which belongs with the fifth postulate of Archimedes, the second proposition being the first as set forth by modern critical writers. Similar variations will be found throughout all of the older editions, but naturally the greatest changes are in the *Stomachion*, the *Method*, and the *Floating Bodies*, all of which have come to light in whole or in newly discovered portions within the last twenty years.

The third justification for a new translation is seen in the general method employed in setting forth the mathematical works of the ancients. Just how far should a translation be literal, and how far should it seek to preserve the form in which the original is cast? The answer to this question depends upon the purpose of the translator. In Heiberg's version he gives the Greek text as it appears in the best manuscripts extant, using the modern forms of the letters and the alphabetic numerals. The text gives, therefore, a correct idea of the original form of statement. In his Latin translation, however, he breaks away from this original form and introduces such modern symbols as appear in the lines

$$\Delta\Gamma + \Gamma B > \Delta B,$$
$$EH : HZ = A\Gamma : A\Theta,$$

and

$$\angle NH\Gamma < 2\,\angle K M.$$

When Heath prepared his edition he did the same; but he went even further, arranging the proofs still more after the manner of the modern textbook maker.

As to which of these methods is to be preferred, the reader's preference must be final. The Heath plan is unquestionably the easier for one who is simply searching for the mathematical principles involved; but the method used by Heiberg in the Greek text is the better if one wishes to attune his mind to that of the Greek scholar. Most students who have seriously studied Euclid and Apollonius and Archimedes, for example

marvel no doubt at the clarity of the reasoning as set forth in rhetorical form, and at the ability of the Greek mind to accomplish, with no special mathematical symbolism, the results that appear in the classical works of these men. For such students the text which most nearly represents the original is the better; but for the general reader, who is not particularly interested in the mental processes of the Greek, the modern arrangement of proofs, with our common symbols, will be the more simple.

In the two books under review the reader will find the original Greek method set forth at its best in the translation of M. Ver Eecke. It is a model of elegant French and of close rendition of the original text. The scholarly world is greatly indebted to him and to his publishers for their "labor of love." A comparison of the French text with random selections from the Heiberg version shows that the translation has been made with elegance and precision.

The version of Dr. Czwalina-Allenstein is less pretentious, being limited to the *Sphere and Cylinder*. It consists of a careful translation of Heiberg's Latin text, with the modern symbols there used. Instead, however, of displaying the equations and inequalities so that the eye grasps them easily, the translator has run them in with the text. This was doubtless due to the desire to save space, although it is not probable that the size of the book is thereby reduced by more than a couple of pages. The result is that we have neither the original form of expression as seen in the Greek version of Heiberg and in M. Ver Eecke's translation, nor the modern display arrangement as seen in Heath's edition. Nevertheless we have every reason to be thankful for such an accurate piece of work and for its publication in a form that makes the *Sphere and Cylinder* available at a nominal price for all German readers. Indeed, it is with some feeling of envy that one considers the advantages that German scholars have over us in the possession of the Ostwald *Klassiker* in such worthy translations.

On the whole, each of the books under review serves a worthy purpose. The smaller of the works serves such a purpose in that it places in the hands of all who are interested in the subject one of the greatest mathematical classics of the Greeks, and at a price which, even in these troublous times, is insignificant. The more pretentious work of M. Ver Eecke serves a more distinguished purpose, but perhaps no more important one, in that it gives us the first complete modern translation of the works of Archimedes, based upon the latest discoveries and the best textual criticism, and in that this translation gives us the form of the original treatises so that we can more easily place ourselves *en rapport* with the mind of the great genius who gave them to the world.

DAVID EUGENE SMITH

A COURSE IN EXTERIOR BALLISTICS

A Course in Exterior Ballistics. Ordnance Textbook. Washington, Government Printing Office, 1921. 127 pp.

This is one of the books prepared by the Ordnance Department, and is issued "for the information and guidance of all concerned." It is based upon a course in ballistic methods given in the Ordnance School of Application at the Aberdeen Proving Ground by Captain R. S. Hoar during the winter of 1919–20. Captain Hoar was at that time in charge of the Ballistic Section of the Proof Department at the Proving Ground. He is the real author of the book.

As this is the first attempt to give, in anything like complete form, an exposition of the new ballistic methods developed during the World War in this country, it will be of unusual interest. The criticisms that appear in this review are all intended to be constructive, and are the result of observations made while using the book as one of several texts in a course given at Aberdeen during the next term of the school referred to above.

In Chapters I–IV the author gives what he considers "the irreducible minimum of higher mathematics" necessary for an understanding of the later chapters. As prerequisites the student is supposed to be grounded thoroughly "in algebra, in trigonometry, and to know enough of calculus to appreciate the meaning of a derivative, a differential, and a definite integral." The irreducible minimum consists of partial differentiation, successive approximations, the effect of differential variations, and finite differences. Following Chapter IV the book takes up in succession the following topics: An introduction to modern ballistic methods; the computation of trajectories; the computation of differential corrections; the construction of range tables. Alternate methods and a brief mention of the more involved mathematical processes are given in supplements at the end of the book. These topics will be reviewed in this order.

Of the first four chapters those on successive approximations and finite differences are written clearly and explain these subjects in sufficient detail for a thorough understanding of these methods in subsequent chapters. But in interpolating to find range, time of flight, etc., Lagrange's interpolation formula would serve the purpose quite as well as the one given, is at least as easy to use, and is more easily understood by the average student. The chapter on partial differentiation could have been omitted as it contains nothing which cannot be found in standard texts on calculus with which the student is supposed to have some familiarity. The chapter on differential variations and the subject of differential corrections as later developed are, it seems to me, decidedly inadequate. In my opinion the author should have shown in detail how Major F. R. Moulton first derived the differential equations that must be satisfied by the differences between the coordinates of a point on the standard and those of a point on the disturbed trajectory, how these were subsequently simplified, and how

Professor Bliss introduced his further modification. The introduction of the system of adjoint equations should be presented step by step. To feel that students should have a drill on partial differentiation before undertaking to read the subsequent chapters with appreciation and to assume, tacitly at least, that nothing more would have to be said about adjoint systems than is given in this book, is somewhat out of proportion. One could not ask for a better introduction to this phase on the subject than is given by Professor Dunham Jackson in his ordnance pamphlet published in August, 1919.

The differential equations of the motion of a projectile are

$$X'' = -EX', \qquad Y'' = -EY' - G,$$

where $E = GH/C$, and primes indicate the order of the derivative with respect to t. Of these G is a tabulated function of the *resistance divided by the velocity*, H is a function of the altitude, and C is the ballistic coefficient. These differential equations as well as those used in calculating differential corrections are integrated by means of numerical integration. The introduction of this process as applied to ballistics in this country is due to Major Moulton. The method may be used in this case to obtain any degree of accuracy provided that the physical data are sufficiently accurate. But actually a trajectory so computed is no more accurate than the tabulated G-function, and its accuracy is further limited by the fact that the behavior of a projectile is only approximately characterized by its ballistic coefficient. The newer methods are such a decided advantage, however, over all earlier ones that their acceptance as the best working basis for practical ordnance engineering is not open to question.

It is unfortunate that the tables used in the computation of trajectories and differential corrections are not available to the average reader. They are regarded as confidential by the U. S. Army. To the reader who does not have access to them, these parts of the book will not be more interesting than the solution of problems in spherical trigonometry would be without the proper tables.

One correction that is made unnecessary by the new methods is that for the curvature of the earth. This is accomplished by the introduction of a system of curvilinear coordinates; that is, the distances x are measured along the curved surface of the earth, and the distances y are measured along the extensions of the radii of the earth. In calling attention to the very great superiority of this method of treating the problem it is unfortunate that the author makes a grave error at the bottom of page 67 in stating that the error due to disregarding the curvature of the earth (that is, by regarding the earth as flat) is the enormous quantity $.078\ R^2 \operatorname{ctn} \omega$, where R is the radius of the earth and ω is the angle of fall. The correct expression is $.000000078\ X^2 \operatorname{ctn} \omega$, where X is the range in meters and ω is the angle of fall.

This error shows that it might have been of considerable advantage to work out a numerical illustration of each correction formula. Such an error would not have occurred if that had been done. Also practical persons like to know how much error is to be expected from a given source.

As another instance of the same thing in Chapter XIII a very interesting treatment of the influence of the rotation of the earth upon a projectile leaves the reader in doubt as to whether he should expect a variation of 50 m. or of 150 m. in a given range.

The most recent investigations in ballistics seem to indicate that the effect of yaw has been underestimated. The author mentions the fact that this has usually been allowed for in the ballistic coefficient and that we have only very inaccurate information on the subject at present. Not only is this true, but very slight inaccuracies in the manufacture of two projectiles might be sufficient to cause yaws in different directions of such magnitude that more variation would be produced from this cause than by disregarding entirely corrections which have been laboriously made.

The methods developed in this book, with some variations and additions, have been used by Dr. A. A. Bennett in the construction of new ballistic tables. It was my privilege to use these in the course at Aberdeen. They are easy to understand, easy to use, and from every standpoint are all that could be desired at the present time by the practical ballistician.

Although it is generally conceded that real and lasting contributions to ballistics are made rarely, the use of numerical integration will always stand out as a definite contribution of this period of interest in the subject. Also a very decided advance has been made in the way differential equations have been used to attack problems in ballistics. The book under review is the pioneer in presenting these methods "for the information and guidance of all concerned," and all subsequent books will follow it to a great extent. It is to be hoped that a revision of the book will soon be made, if for no other reason, because of the large number of typographical errors. In spite of the very greatest industry and patience on the part of the author a number of these are still to be found, and the text is manifestly unfair to the author in its present form.

It is not an easy matter to carry through successfully the publication of such a book, especially if one is an army officer and subject to the commands of a superior who may or may not appreciate the character and importance of the work. The Army is fortunate, I think, in having had a man of the author's training, experience, enthusiasm, and ability to get things done, to undertake and carry through such an important contribution to the science of ballistics. Full credit is given to all of the mathematicians whose work forms a part of the text, and it would be unappreciative indeed if full credit were not given to the author for any good result that the publication of the book has had or will have in the future. No course in ballistics would be complete without its use as a text, but it should be used to supplement the old ballistics of Ingalls rather than to replace it. The book cannot be read with facility by the average graduate of West Point because he is unfamiliar with much of the mathematics. It might have the very excellent result of raising the standard of mathematics as taught at West Point.

J. E. Rowe

SHORTER NOTICES

Leçons sur l'Intégration des Équations aux Dérivées Partielles du Premier Ordre. 2d edition, revised and enlarged. By Edouard Goursat. Paris, J. Hermann, 1922. 459 pp.

Almost a generation has passed since the first edition of this book appeared in 1890. It was not reviewed in the BULLETIN, for the BULLETIN itself had not yet appeared above the mathematical horizon. But during all of this time it has been an invaluable work of reference. The first edition was worked out by the author's pupil C. Bourlet from a course of lectures given by Goursat. The present edition has been entirely rewritten by Goursat himself. While the general outline and scope of the book has not been altered, the chapters dealing with existence theorems and linear equations have been entirely remodelled. All of the other chapters have been revised, and though they contain much that is the same as in the first edition, they have been somewhat rearranged and enlarged so that the present volume contains 459 pages as against 354 pages in the first edition.

A comparison of the two editions is very interesting. One finds in the present edition a maturity of thought, a ripeness of judgment, and a touch of the master, which is not present in the first edition. Nevertheless there is a freshness of style about the first edition, and a simplicity that might well recommend it to a younger student. In the preface to this edition Goursat states that he has not taken up the method of Pfaff as he intends to devote an entire volume to the *Problem of Pfaff*, which he hopes will appear soon.

The paper in the copy which we have for review leaves much that is to be desired. It is of poor quality, dark in color, and not of a uniform tint. It reflects, doubtless, the present economic situation.

W. D. MACMILLAN

Darstellende Geometrie. By Th. Schmid, associate professor of geometry at the technical school of Vienna. Volume I, third edition. (Sammlung Schubert, LXV.) Berlin and Leipzig, Vereinigung wissenschaftlicher Verleger, 1922. 283 pages and 170 figures.

The large demand for this popular book has necessitated a third edition of the first volume within three years after the appearance of the second. The characteristic features have been retained, and five new paragraphs added; these are a method of measuring the eccentricity of a conic from a skeleton drawing, its use in putting a cone of revolution through the conic, an approximate construction of π, a determination of the pitch of a regular helix, and a discussion of the existence of an isolated bitangent in the projection of a shadow on a cylinder. Several more historical remarks have been added, and practically all the few typographical errors of the earlier editions corrected.

VIRGIL SNYDER

Elastizitätslehre für Ingenieure, I. *Grundlagen und Allgemeines über Spannungszustände, Zylinder, Ebene Platten, Torsion, Gekrümmte Träger*. By Prof. Dr.-Ing. Max Ensslin. (Sammlung Göschen, Nr. 519.) Zweite Auflage. Berlin und Leipzig, Vereinigung wissenschaftlicher Verleger, 1921. 146 pp.

This little book forms one of that admirable collection of booklets known under the general title of SAMMLUNG GÖSCHEN, and it is a splendid representative of that collection. The problem of the technical theory of elasticity is to show how the forces acting in a given case determine the materials and the form of engineering construction, and to give the means of measurement whereby a breakage is avoided, material is saved, and the form of the unit constructed is held within allowable limits. Engineers have sought to meet the problem partly by a study of experimental data, and they have devised simple approximation formulas and simple graphic methods sufficiently accurate to meet the demands for practical safety and for a reasonable cost of construction. Naturally, the technician soon feels the need of a more accurate method of computing the form and the weight of his materials and the cost of construction; he also desires a proof of his approximation theory. He finds these needs met by the study of the mathematical theory of elasticity; for not only does this latter theory give him valuable insight into the fundamental principles of stresses and strains due to the presence of forces, and into the conditions for the occurrence of the elastic limit, but it also furnishes him many simple formulas useful in computation.

It is the purpose of this and its companion volume to serve as an introduction into these fields. The present volume treats of the general theory of stresses, cylinders, flat plates, the torsion of straight rods, and the deformation of curved rods. Keeping in mind the needs of the technician, the author has applied the theoretical results to many special cases arising in practice, has worked many numerical examples, and has given the graphic representation of the stresses involved in his problems.

JOSEPH LIPKA

Business and Investment Forecasting. By Ray Vance. New York, Brookmire Economic Service, 1922. 132 pp.

A small paragraph must suffice for this modest, unpretentious, but very delightful book. Nowadays business and investment barometers built upon the methods of correlation are not unknown to those who have no idea of scientific statistical methods. Vance's aim is to give the intelligent but not technically informed reader a general notion of the history of such barometers and of the principles of their construction. He appreciates and explicitly points out the great difficulties and uncertainties connected with statistical forecasting, the need for constant revision, for wariness, and for hard study. The main emphasis is naturally laid on the Brookmire Barometers, but there are no hard words for others. Anybody who will read Mr. Vance's little book will gain not only an insight into its subject matter, but will feel a high degree of confidence in the integrity of any barometer constructed by the author.

EDWIN B. WILSON

Einführung in die Theorie der Gewöhnlichen Differentialgleichungen auf Funktionentheoretischer Grundlage. By Ludwig Schlesinger. Dritte, neubearbeitete Auflage. Berlin, Vereinigung wissenschaftlicher Verleger, 1922. 8 + 326 pp.

The first edition of this work appeared in 1900 under the title *Einführung in die Theorie der Differentialgleichungen mit einer unabhängigen Variablen*, and was reviewed for the BULLETIN by Maxime Bôcher. The second edition appeared in 1903 and was merely a reprint of the first. The present edition, however, is thoroughly revised and enlarged, and merits the new title which it has received, for even the first edition was written from the point of view of the theory of functions of a complex variable.

Scarcely half of the original work reappears in the present edition, and even that has been rearranged and more thoroughly germanized by the substitution of words of purely germanic origin for words of a latin origin For example, *Problem* is erased and *Aufgabe* is substituted, *determinieren* is replaced by *bestimmen*, *Relationen* by *Beziehungen*, etc. This part includes existence theorems for integrals of differential equations of the first order and the discussion of Gauss' and Bessel's equations.

The introduction of the original edition has been expanded into a chapter interpreting the differential equation for real values of the variables. The discussion of linear equations has been simplified by the use of the calculus of matrices. There is a good discussion of differential equations of order higher than the first with fixed critical points, and in the last chapter even the linear integral equations have been touched upon.

All writers upon differential equations insist upon the importance of their subject from the point of view of the applications, and the present writer is no exception. It seems strange therefore that all of them should maintain such a profound silence upon the method, generally known as mechanical quadrature, of following a particular solution in numerical cases. It has been in use, particularly among the astronomers, for a hundred years, and in many cases gives us all of the information which we desire when analytic methods fail. It is much as though writers on the theory of algebraic equations should conspire to ignore Horner's Method and other processes which show us how to find the roots in numerical cases. From the point of view of applications, it merits more attention.

W. D. MACMILLAN

La Matière et l'Énergie selon la Théorie de la Relativité et la Théorie des Quanta. By Louis Rougier. Nouvelle édition, revue et augmentée. Paris, Gauthier-Villars, 1921. 112 pp.

A notice of an English translation, by M. Masius, of the earlier edition of this book has appeared in this BULLETIN (vol. 28, p. 319). As the translator had access to the revised version of the first edition, it is, essentially, the same book. The author appears too ready to accept extreme views with respect both to the theory of relativity and the quantum theory, without subjecting them to a critical survey. On the other hand, he has given a very lucid account of the way in which our conceptions of the interaction of matter and radiant energy have been undergoing a change.

E. P. ADAMS

NOTES

The concluding number of volume 22 of the TRANSACTIONS OF THIS SOCIETY (October, 1921, published in May, 1923) contains the following papers: *On certain numerical invariants of algebraic varieties with application to Abelian varieties (continued)*, by S. Lefschetz; *On restricted systems of higher indeterminate equations*, by E. T. Bell; *Maximum modulus of some expressions of limited analytic functions*, by S. Kakeya; *Differential variations in ballistics, with applications to the qualitative properties of the trajectory*, by T. H. Gronwall; *An expansion theorem for a system of linear differential equations of the first order*, by W. A. Hurwitz.

The concluding number of volume 44 of the AMERICAN JOURNAL OF MATHEMATICS (October, 1922) contains: *On the kernel of the Stieltjes integral corresponding to a completely continuous transformation*, by C. A. Fischer; *Equivalence and reduction of pairs of Hermitian forms*, by M. I. Logsdon; *Plane cubics with a given quadrangle of inflexions*, by B. M. Turner; *Normal ternary continued fraction expansions for the cube roots of integers*, by P. H. Daus; *Concerning compact Kürschák fields*, by V. D. Gokhale.

The third number of volume 23, series 2, of the ANNALS OF MATHEMATICS (March, 1922) contains: *Dirichlet's problem*, by G. E. Raynor; *Annihilators of modular invariants and covariants*, by O. C. Hazlett; *Systems of linear inequalities*, by W. B. Carver; *Euler squares*, by H. F. MacNeish; *Geometric aspects of Einstein's theory*, by J. Pierpont; *Cauchy's paper of* 1814 *on definite integrals*, by H. J. Ettlinger; *Arithmetical deduction of Kronecker's class-number relations*, by G. H. Cresse; *Cyclotomic heptasection for the prime* 43, by P. O. Upadhyaya; *Summation of a double series*, by T. H. Gronwall.

The following members of the Society have been requested to represent the Committee on Endowment in their respective institutions and in the surrounding territory: C. H. Ashton, Harry Bateman, E. T. Bell, H. F. Blichfeldt, W. C. Brenke, Daniel Buchanan, Paul Capron, D. R. Curtiss, Arnold Dresden, W. H. Echols, John Eiesland, G. C. Evans, J. C. Fields, Tomlinson Fort, Harris Hancock, James Harkness, M. W. Haskell, H. E. Hawkes, E. R. Hedrick, Dunham Jackson, N. J. Lennes, P. H. Linehan, W. R. Longley, J. L. Markley, G. A. Miller, W. E. Milne, Frank Morley, R. K. Morley, G. D. Olds, R. G. D. Richardson, H. L. Rietz, F. H. Safford, G. E. F. Sherwood, H. E. Slaught, C. E. Smith, R. B. Stone, K. D. Swartzel, J. H. Tanner, H. W. Tyler, Oswald Veblen, H. S. White, J. W. Young.

The following important books have been published recently, or are announced for early publication, by American university presses: *Relativity and Modern Physics*, by G. D. Birkhoff, Harvard University Press, 1923; *The Meaning of Relativity*, by Albert Einstein, and *Transformations of Surfaces*, by L. P. Eisenhart, Princeton University Press, 1923 (the latter

being published with the cooperation of the National Research Council); *Lectures on Cauchy's Problem in Linear Partial Differential Equations*, by Jacques Hadamard, announced by the Yale University Press for early publication.

The Berlin Academy of Sciences announces the early publication of volume 7 of Weierstrass's Works.

The following honorary doctorates have been conferred: by the University of Edinburgh, on Professor H. F. Baker, of St. John's College, Cambridge; by the University of Aberdeen, on Professor E. W. Hobson, of Christ's College, Cambridge; by the University of Pennsylvania, on Sir Joseph Thomson, master of Trinity College.

The tricentenary of the birth of Blaise Pascal, which occurs on June 19, 1923, will be commemorated by celebrations to be held on July 8–9, under the auspices of the French government and the Paris Academy of Sciences.

Professor A. A. Michelson, of the University of Chicago, has been elected president of the National Academy of Sciences. Professors E. W. Brown, of Yale University, Max Mason, of the University of Wisconsin, and D. L. Webster, of Stanford University, have been elected to membership in the Academy.

Professor A. A. Michelson has been elected first vice-chairman of the National Research Council. Professor M. I. Pupin, of Columbia University, has been elected a member at large of the executive board.

Sir Ernest Rutherford, of Cambridge University, has been elected an honorary foreign member of the Washington Academy of Sciences.

The price of the JAHRBUCH ÜBER DIE FORTSCHRITTE DER MATHEMATIK has been substantially reduced. Beginning with volume 46, the first number of which has just been published, the price will be about one cent a page, with a reduction of twenty-five per cent to members of the American Mathematical Society and the Mathematical Association of America. This reduced price is practically the same as the pre-war price. To take advantage of it, orders should be sent directly to Professor L. Bieberbach, Berlin-Schmargendorf, Marienbaderstrasse 9.

Professor F. S. Carey has resigned from the chair of mathematics at the University of Liverpool, after thirty-seven years of service.

Professor M. J. M. Hill, of University College, London, has resigned.

At the University of Grenoble, M. Gosse has been promoted to a professorship of mathematics.

Dr. H. M. Dadourian, associate professor of physics at Trinity College, Hartford, has been appointed Seabury professor of mathematics.

Dr. Louis Weisner, of Columbia University, has been appointed instructor in mathematics at the University of Rochester.

Mr. J. B. Rosenbach, instructor in mathematics at the Carnegie Institute of Technology, has been made assistant professor of mathematics.

Mr. O. H. Rechard, Jr., instructor in mathematics at the University

of Wisconsin, has been appointed assistant professor of mathematics at the University of Wyoming.

Professor J. L. Riley has resigned his professorship of mathematics at John Tarleton Agricultural College, Stephenville, Texas.

At Rutgers College, Professor A. A. Titsworth, formerly in the department of Civil Engineering, has been transferred to the department of mathematics as a full professor. Mr. W. B. Campbell has been appointed instructor.

Miss Ruth Thompson has been appointed instructor in mathematics at the New Jersey College for Women.

The following appointments have been made in the Department of Mathematics at Harvard: Philip Franklin, Benjamin Peirce instructor; H. L. Garabedian, K. W. Halbert, R. L. Jeffrey, E. R. C. Miles, F. W. Perkins, M. M. Slotnick and T. L. Smith, instructors.

At Ohio Northern University, Mr. J. T. Fairchild has been promoted to a professorship of mathematics.

Rev. Paul Muehlman, formerly head of the department of mathematics at Marquette University, has been transferred to Loyola University, Chicago, as head of the department there.

Mr. B. F. Kimball has been appointed instructor in mathematics at Tulane University.

At the Texas Agricultural and Mechanical College, Assistant Professors D. C. Jones and W. L. Porter have been promoted to associate professorships of mathematics, Mr. W. L. Hughes and Mr. P. K. Smith have been appointed assistant professors, and Mr. C. E. McCurry an instructor.

At the University of Denver, Professor G. W. Gorrell of the Colorado State School of Mines has been appointed an associate professor, and Dr. E. Frances Seiler of the University of Illinois an instructor.

Professor Matyáš Lerch, formerly of the University of Freiburg, Switzerland, and since 1906 of the Polytechnic Institute at Brno, died August 3, 1922, at the age of sixty-two. The Academy of Sciences of the Institute of France awarded to Professor Lerch in 1900 the grand prize for mathematical sciences.

Professor Čenek Jarolímek, of the Polytechnic Institute of Prague, died December 14, 1921, at the age of seventy-five.

Professor J. D. van der Waals, of the University of Amsterdam, died March 8, 1923, at the age of eighty-five.

Dr. J. G. Leathem, fellow and bursar of St. John's College, Cambridge, coeditor with Professor E. T. Whittaker of the Cambridge Mathematical Tracts, died March 19, 1923, at the age of fifty-one.

Sir John Venn, president of Gonville and Caius College and for many years lecturer in logic and philosophy at Cambridge University, died April 4, 1923, at the age of eighty-eight.

Professor A. G. Webster, of Clark University, died May 15, 1923, at the age of fifty-nine. He had been a member of this Society since 1891.

NEW PUBLICATIONS

PART I. PURE MATHEMATICS

BOEHM (K.). Begriffsbildung. Karlsruhe, G. Braun, 1922. 46 pp.

CREAK (T. G.). See CUNNINGHAM (A.).

CRESSE (G. H.). See DICKSON (L. E.).

CUNNINGHAM (A.), WOODALL (H. J.) and CREAK (T. G.). Haupt-exponents, primitive roots, and standard congruences. London, F. Hodgson, 1922. 136 pp.

DICKSON (L. E.). History of the theory of numbers. Volume III: Quadratic and higher forms. With a chapter on the class number, by G. H. Cresse. Washington, Carnegie Institution, 1923. 8vo. 4 + 313 pp.

EISENHART (L. P.). Transformations of surfaces. Published with the cooperation of the National Research Council. Princeton, University Press, 1923. 10 + 380 pp. $4.00

EPSTEIN (P.). See WEBER (H.).

FETTWEIS (E.). Wie man einstens rechnete. (Mathematisch-Physikalische Bibliothek, Nr. 49.) Leipzig, Teubner, 1923. 56 pp.

JUNKER (F.). Höhere Analysis. I: Differentialrechnung. 3te, verbesserte Auflage. (Sammlung Göschen.) Berlin, Vereinigung wissenschaftlicher Verleger, 1921. 8vo. 204 pp.

MINER (J. R.). Tables of $\sqrt{1 - r^2}$ and $1 - r^2$ for use in partial correlation and in trigonometry. Baltimore, Johns Hopkins Press, 1922. 8vo. 49 pp.

MÜLLER (C.). Mathematik. In Vom Altertum zur Gegenwart. Die Kulturzusammenhänge in den Hauptepochen und auf den Hauptgebieten. Skizzen. 2te, vermehrte Auflage. Leipzig, Teubner, 1921. 10 + 386 pp.

OTZEN (R.). See TIMERDING (H. E.).

SCHILLER (L.). Untersuchungen über laminare und turbulente Strömung. (Leipziger Habilitationsarbeit.) Berlin, Verlag des Vereines deutscher Ingenieure, 1922. 36 pp.

STEELE (R.). The earliest arithmetics in English. Edited, with an introduction, by R. Steele. London, Oxford University Press, 1923. 8vo. 18 + 84 pp.

TIMERDING (H. E.). Mathematik. (Handbibliothek für Bauingenieure, herausgegeben von R. Otzen, Band I, Teil I.) Berlin, Springer, 1923. 8 + 242 pp.

WEBER (H.) und WELLSTEIN (J.). Enzyklopädie der Elementarmathematik. 1ter Band: Arithmetik, Algebra und Analysis. 4te Auflage, neubearbeitet von Paul Epstein. Leipzig, Teubner, 1922. 16 + 568 pp.

WELLSTEIN (J.). See WEBER (H.).

WOODALL (H. J.). See CUNNINGHAM (A.).

PART II. APPLIED MATHEMATICS

ALLIATTA (G.). Die Radioaktivität im Weltbild der Aethermechanik. Leipzig, Hillmann, 1922. 8vo. 14 pp.

ANDOYER (H.). Cours de mécanique céleste. Tome I. Paris, Gauthier-Villars, 1923.

BIRKHOFF (G. D.). Relativity and modern physics. Cambridge, Mass., Harvard University Press, 1923.

BOHR (N.). Atomernes Bygning og Stoffernes fysiske og kemiske Egenskaber. Kjøbenhavn, Gjellerup, 1922. 72 pp.

BORN (M.). La théorie de la relativité d'Einstein et ses bases physiques. Traduit sur la 2e édition allemande par F. A. Finkelstein et J. G. Verdier. Paris, Gauthier-Villars, 1923. 8vo. 339 pp.

BOTTLINGER (K. F.). See STRÖMGREN (E.).

BOUTARIC (A.). La vie des atomes. Paris, Flammarion, 1923. 16mo. 248 pp.

BRILLOUIN (L.). La théorie des quanta et l'atome de Bohr. Paris, Les Presses Universitaires de France, 1922. 8vo. 181 pp.

———. See FABRE (L.)

DE BROGLIE (M.). Les rayons X. Paris, Les Presses Universitaires de France, 1922. 8vo. 164 pp.

BURTON (E. F.). The physical properties of colloidal solutions. 2d edition. London, Longmans, 1921. 8vo. 220 pp.

COURANT (R.). See MADELUNG (E.).

CURRY (W. A.). See MORECROFT (J. H.).

DINWOODIE (W.). Wave power transmission. London, Spon, 1923. 8vo. 28 pp.

DREYER (G.). Formeln, Begriffserklärungen und Lehrsätze aus der reinen und angewandten Festigkeitslehre. 3te Auflage. Leipzig, Jänecke, 1922. 8 + 96 pp.

FABRE (L.). Les théories d'Einstein. Nouvelle édition, épurée, accrue de notes liminaires, d'un exposé des théories de Weyl, et de trois notes de MM. Guillaume, Brillouin, et Sagnac sur leurs propres idées. Paris, Payot, 1922. 8vo. 255 pp.

FICHOT (E.). Les marées et leur utilisation industrielle. Paris, Gauthier-Villars, 1922. 8vo. 256 pp.

FINKELSTEIN (F. A.). See BORN (M.).

FISCHER (V.). Eine Darstellung des Nernst'schen Wärmetheorems. Teil 1. Frankfurt, Blazek und Bergmann, 1923. 8vo. 23 pp.

FLEURY (H.). Exposé élémentaire de la théorie d'Einstein. Paris, E. Larose, 1922. 8vo. 34 pp.

FONTENÉ (G.). La relativité restreinte, avec un appendice sur la relativité généralisée. Paris, Vuibert, 1923. 12mo. 116 pp.

FOOTE (P. D.) and MOHLER (F. L.). The origin of spectra. (American Chemical Society Monograph Series.) New York, Chemical Catalogue Company, 1922. 8vo. 250 pp.

FUCHS (F.). Elektrische Strahlen und ihre Anwendung (Röntgentechnik). München und Berlin, Oldenbourg, 1922. 35 pp.

GLOVER (J. W.). Tables of applied mathematics in finance, insurance, statistics. Ann Arbor, George Wahr, 1923. 676 + 13 pp. $4.50

GORDAN (J. W.). Generalized linear perspective; treated with special reference to photographic land surveying and military reconnaissance. London, Constable, 1922. 16 + 184 pp.

GRAFF (O.). See GROLL (M.).

GROLL (M.). Kartenkunde. II: Der Karteninhalt. 2te Auflage, neubearbeitet von O. Graff. (Sammlung Göschen.) Berlin, Vereinigung wissenschaftlicher Verleger, 1923. 133 pp.

GUILLAUME (—.). See FABRE (L.).

LÖFFLER (S.) und RIEDLER (A.). Reibungstriebwerke und ihre Missdeutung durch Theoretiker. München und Berlin, Oldenbourg, 1921.

MADELUNG (E.). Die mathematischen Hilfsmittel des Physikers. (Die Grundlagen der mathematischen Wissenschaften in Einzeldarstellungen, herausgegeben von R. Courant, Band 4.) Berlin, Springer, 1923. 12 + 247 pp.

MAREC (E.). Les enroulements industriels des machines à courant continu et à courants alternatifs. Théorie et pratique. 2e édition. Paris, Gauthier-Villars, 1921. 8vo. 9 + 240 pp.

VON MISES (R.). Fluglehre. 2te, durchgesehene Auflage. Berlin, Springer, 1922. 8 + 210 pp.

MOHLER (F. L.). See FOOTE (P. D.).

MORECROFT (J. H.), PINTO (A.) and CURRY (W. A.). Principles of radiocommunication. New York, Wiley, and London, Chapman and Hall, 1921. 10 + 935 pp.

PINTO (A.). See MORECROFT (J. H.).

RIEDLER (A.). See LÖFFLER (S.).

SAGNAC (—.). See FABRE (L.).

SCHMIDT (H.). Das Weltbild der Relativitätstheorie. 3te Auflage. Hamburg, Hartung, 1922. 8vo. 8 + 139 pp.

SCHWINGE (O.). Eine Lücke in der Terminologie der Einsteinschen Relativitätslehre. Berlin, Commissionsverlag Puhle, 1921. 30 pp.

STEPHAN (P.). Die technische Mechanik des Maschineningeneurs. Band 1: Allgemeine Statik. Band 2: Die Statik der Maschinenteile. Band 3: Bewegungslehre und Dynamik fester Körper. Berlin, 1921–22.

STOCK (A.). The structure of atoms. Translated from the second German edition by S. Sugden. Revised and enlarged. London, Methuen, 1923. 8 + 88 pp.

STRÖMGREN (E.). Astronomische Miniaturen. Aus dem Schwedischen übersetzt von K. F. Bottlinger. Berlin, Springer, 1922. 8 + 88 pp.

SUGDEN (S.). See STOCK (A.).

TURNER (L. B.). Wireless telegraphy and telephony. Cambridge, University Press, 1921. 8vo. 14 + 196 pp.

VERDIER (J. G.). See BORN (M.).

WALDEN (P.). Optische Umkehrerscheinungen (Waldensche Umkehrung). Braunschweig, Vieweg, 1919. 5 + 214 pp.

WARNANT (L.). Les théories d'Einstein. Essai de réfutation. Examen critique. Paris, Alcan, 1922. 12mo. 144 pp.

JOURNALS OF THIS SOCIETY

Increases in Prices and Present Rates on Back Volumes

(Subject to change without notice.)

On account of the reduction of stock due to unusual sales, the prices of volumes 13 and 18 of the TRANSACTIONS, and of volume 18 of the BULLETIN have been increased, and volume 11 of the BULLETIN has been withdrawn from sale except with complete sets of volumes 11–26 of the BULLETIN. The present prices, subject to change without further notice, are as follows:

I. THE TRANSACTIONS.

1. COMPLETE SETS. Full Set, volumes 1–21, published at $105.00.....$65.00

2. SEPARATE VOLUMES.

Volumes 1, 2, 3, 4, 5, 16, 17, 19, 20, 21, each.....$2.75
Volumes 6, 7, 8, 11, 12, 14, 18, each..... 3.25
Volumes 9, 10, 13, 15, each..... 3.75
Volume 22 and beyond..... 7.00

(Members of the Society and bookdealers may buy vol. 22 and beyond at the usual discounts; but no discount can be allowed on the other prices quoted.)

II. THE BULLETIN.

(Members are requested to donate old numbers, particularly of volumes 1–17, to the Society. We need most seriously vols. 1–10, No. 1 of vol. 11, No. 3 of vol. 14, No. 4 of vol. 15, and No. 2 of vol. 17.)

SEPARATE VOLUMES.

Volumes 1–10.....not for sale
Volumes 11–26 (with index of vols. 11–20.....$45.00
Volumes 12, 13, 16, 18, each..... 3.50
Volumes 19, 20, 21, 22, 23, 24, 25, 26, each 2.50
Volumes 11, 14, 15, 17 for sale only with volumes 11–25
Volume 27 and beyond..... $7.00

(Members of the Society and bookdealers may buy vol. 27 and beyond at the usual discounts; but no discount can be allowed on the other prices quoted.)

COMMITTEE ON SALES OF PUBLICATIONS.

A. B. Coble, E. R. Hedrick, W. R. Longley (*Chairman*).

Those "Cut and Dried" Phrases

You know them all—we all have to read them: "Correspondence Solicited," "Estimates Furnished," "Quality Printing," "First-class Work," etc. Why should they be paraded before your eyes continually when we know that if you have an order for books, monographs, announcements, or any kind of commercial work—*no matter how small*—you will give it to the printer who you feel confident will give you the best result that men and material can produce.

Lancaster Press, Incorporated

Formerly

THE NEW ERA PRINTING CO.,

Incorporated

Lancaster, Pennsylvania

Publications of this Society

Transactions of the American Mathematical Society. Published quarterly. Subscription price for annual volume, $7.00; to members of the Society, $5.25.

Mathematical Papers of the Chicago Congress, 1893. Price, $4.00; to members, $2.00.

Evanston Colloquium Lectures, 1893. By FELIX KLEIN. Price, $1.25; to members, 75 cents.

Boston Colloquium Lectures, 1903. By H. S. WHITE, F. S. WOODS and E. B. VAN VLECK. Price, $2.75; to members, $2.25.

Princeton Colloquium Lectures, 1909. By G. A. BLISS and EDWARD KASNER. Price, $2.50; to members, $2.00.

Madison Colloquium Lectures, 1913. By L. E. DICKSON and W. F. OSGOOD. Price, $2.50; to members, $2.00.

Cambridge Colloquium Lectures, 1916. Part I. By G. C. EVANS. Price, $2.00; to members, $1.50.

Part II. By OSWALD VEBLEN. Price, $2.00; to members, $1.50. Parts I-II, bound together, in cloth. $3.50; to members, $3.00.

Portrait of Maxime Bôcher. Price, 20 cents. **Portrait of F. N. Cole.** Price, 20 cents.

Address all orders to 501 West 116th Street, New York, N. Y.

JOURNAL DE MATHÉMATIQUES PURES ET APPLIQUÉES

Founded in 1836 by Joseph Liouville

EDITED BY
HENRI VILLAT
WITH THE ASSISTANCE OF
R. DE MONTESSUS DE BALLORE and E. PICARD

This Journal, which has been among the foremost exponents of mathematics for nearly a century, was in danger of discontinuance owing to the changed financial situation caused by the war. By prompt action of a group of French mathematicians, its publication has been assured for the immediate future. This group contains, in addition to the editors, such eminent men as Appell, Borel, Boussinesq, Brillouin, Cartan, Drach, Goursat, Guichard, Hadamard, Koenigs, Lebesgue, Montel, Painlevé, Vessiot.

Papers will be accepted and printed in English as well as in French. Among American contributors to recent or to forthcoming numbers may be mentioned Professors Dickson, Eisenhart, Hancock, Wilczynski. It is hoped that the Journal will become an indispensable element in international mathematical relations.

To insure the permanence of the Journal, subscriptions are invited. The present price is 90 francs, now equivalent to about $6.50. Subscriptions or contributions may be sent to the Editor of this BULLETIN or to H. Villat, 11 rue de Maréchal Pétain, Strasbourg, France.

WILEY BOOKS

Just Published

Mathematical Theory of Finance

By **THOMAS MILTON PUTNAM, Ph.D.,**

Professor of Mathematics and Dean of the Undergraduate Division, University of California.

Designed to cover a half-year college course, and intended primarily for students of Commerce. The fundamental principles of compound interest and annuities are included, with applications of the latter to bonds, sinking funds, and amortization, and an introduction to the principles of life insurance. The equivalent of two years of high school Algebra is presupposed and an ability to use logarithms such as may be obtained in the usual elementary courses in Trigonometry.

About 250 carefully selected problems are given, together with interest and annuity tables for purposes of computation.

120 pages. 5¼ by 8. 3 figures. Cloth, $1.75 postpaid.

Other Recently Published Books

Plane Trigonometry

By **ARNOLD DRESDEN, Ph.D.,**

Assistant Professor of Mathematics in the University of Wisconsin.

110 pages. 5½ by 8½. 58 figures. Cloth, $1.60 postpaid.

First Course in the Theory of Equations

By **LEONARD EUGENE DICKSON, Ph.D.,**

Professor of Mathematics, University of Chicago.

168 pages. 6 by 9. 26 figures. Cloth, $1.75 postpaid.

Send for copies for examination.

JOHN WILEY & SONS, Inc.

432 Fourth Avenue, New York

London: Chapman & Hall, Ltd. Quebec: Renouf Publishing Co.

OFFICIAL COMMUNICATIONS

Meetings of the Society have been fixed at the following times and places:

THE SUMMER MEETING OF THE SOCIETY, at VASSAR COLLEGE, September 6–8, 1923.

Abstracts must be in the hands of the Secretary of the Society not later than August 8.

THE SAN FRANCISCO SECTION, at LOS ANGELES, September 17–20, 1923.

Abstracts must be in the hands of the acting Secretary of the Section, D. N. Lehmer, not later than July 26.

NEW YORK CITY, October 27, 1923.

Abstracts must be in the hands of the Secretary of the Society not later than October 13.

THE SOUTHWESTERN SECTION, at COLUMBIA, MISSOURI, December 1, 1923.

Abstracts must be in the hands of the Secretary of the Section, E. B. Stouffer, not later than November 14.

R. G. D. RICHARDSON,
Secretary of the Society.

COLLECTED WORKS OF LEONARD EULER

On the two-hundredth anniversary of the birth of Euler, a committee of the *Society of Swiss Naturalists* launched the project of international cooperation for the publication of his collected works. Academies and individuals subscribed for about 300 sets, and about one hundred thousand (100,000) Swiss francs were collected, for the most part in Switzerland; the American Mathematical Society subscribed five thousand (5,000) francs. Eighteen (18) of the estimated seventy (70) volumes have been published.

By reason of the European War nearly one-half of the subscribers have been unable to meet their obligations in full. Under these circumstances, a considerable number of new subscribers must be secured if the completion of the undertaking is to be possible in the near future. There are three plans for subscribing. Those libraries or individuals wishing for information with a view to joining in promoting this great international undertaking, should communicate with the Official Representative of the Euler Committee for the United States and Canada,

R. C. ARCHIBALD,
Brown University,
Providence, R. I.

Whole Number 315

CONTENTS

PAGE

Incorporation of the American Mathematical Society. By R. G. D. RICHARDSON 241

Report on Curves Traced on Algebraic Surfaces. By S. LEFSCHETZ 242

Note on the Convergence of Weighted Trigonometric Series. By DUNHAM JACKSON 259

Proof of a Formula for an Area. By H. E. BRAY 264

Mathematics for the Layman. By J. W. YOUNG 271

Two Translations of Archimedes. By D. E. SMITH 274

A Course in Exterior Ballistics. By J. E. ROWE 277

Shorter Notices 280

Notes 283

New Publications 286

Any communication intended for this BULLETIN may be addressed to the American Mathematical Society, Prince and Lemon Streets, Lancaster, Pa., or may be sent to separate officials of the Society as follows:

Articles for insertion in the BULLETIN should be addressed to E. R. HEDRICK, Editor of the BULLETIN, 304 Hicks Ave., Columbia, Mo. Reviews should be sent to J. W. YOUNG, Dartmouth College, Hanover, N. H. Notes should be sent to R. W. BURGESS, Brown University, Providence, R. I.

Subscriptions to the BULLETIN, orders for back numbers, and inquiries in regard to non-delivery of current numbers should be addressed to The American Mathematical Society, 501 West 116th Street, New York.

Advertising space is available in the BULLETIN at $16 per page, $9 per half-page, $5 per quarter page, per issue. Address correspondence regarding advertising to H. L. RIETZ, University of Iowa, Iowa City, Iowa.

Changes of address of members, exchanges, and subscribers should be communicated at once to the Secretary of the Society, R. G. D. RICHARDSON, 501 West 116th Street, New York.

The initiation fees ($5.00) and annual dues ($6.00) of members of the Society are payable to the Treasurer of the Society, W. B. FITE, 501 West 116th Street, New York.

LANCASTER PRESS, INC., LANCASTER, PA.